Queuing Theory

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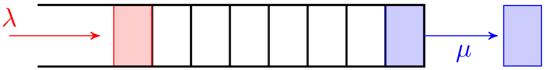
Queues

- Queuing theory is concerned with the (boring) issue of waiting
  - $\Rightarrow$ Waiting is boring, queuing theory not necessarily so
- "Customers" arrive to receive "service" by "servers"
  - $\Rightarrow$ Between arrival and start of service wait in queue
- Quantities of interest (for example)
  - $\Rightarrow$ Number of customers in queue $\Rightarrow L$ (for length)
  - $\Rightarrow$ Time spent in queue $\Rightarrow W$ for (wait)
- Queues are a pervasive application of CTMCs

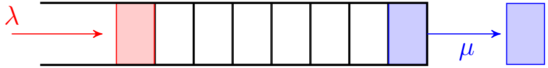


Where do queues appear?

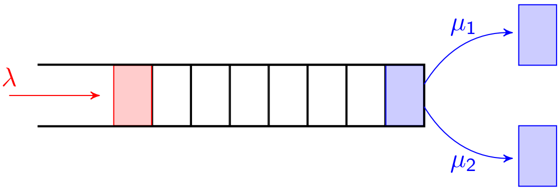
- Queues are fundamental to the analysis of (public) transportation
  - Wait to enter a highway $\Rightarrow$ Customers = cars
  - Q: Subway travel times, subway or buses?
  - Q: Infrequent big buses or frequent small buses?
- Packet traffic in communication networks
  - Route determination, congestion management
  - Real-time requirements, delays, resource management
- Logistics and operations research
  - Customers = raw materials, components, final products
  - Customers in queue = products in storage = inactive capital
- Customer service
  - Q: How many representatives in a call center? Call center pooling

Examples of queues

- Simplest rendition $\Rightarrow$ Single queue, single server, infinite spots
  - $\Rightarrow$ Simpler if arrivals and services are Poisson $\Rightarrow$ M/M/1 queue
  - $\Rightarrow$ Limiting number of spots not difficult $\Rightarrow$ Losses appear



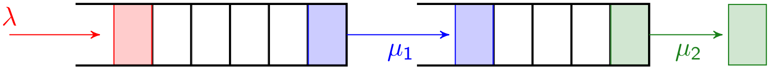
- Multi-server queues $\Rightarrow$ Single queue, many servers
  - $\Rightarrow$ M/M/c queue $\Rightarrow$ c Poisson servers (i.e., exp. service times)



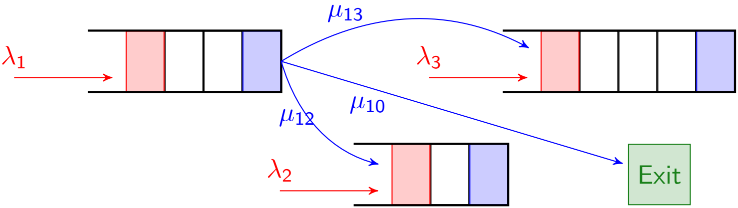
Networks of queues

- Groups of interacting queues $\Rightarrow$ Applications become interesting

Ex: A queue tandem



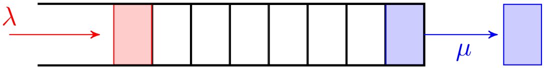
- Can have arrivals at different points and random re-entries



- Batch service and arrivals, loss systems (not considered)

M/M/1 queue

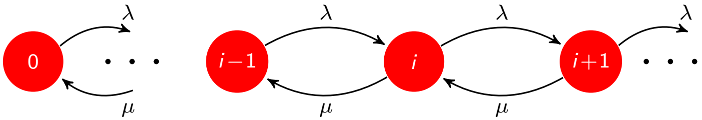
- Arrival and service processes are Poisson $\Rightarrow$ Birth & death process
  a) Customers arrive at an average rate of λ per unit time
  b) Customers are serviced at an average rate of μ per unit time
  c) Interarrival and inter-service time are exponential and independent



- Hypothesis of Poisson arrivals is reasonable
- Hypothesis of exponential service times not so reasonable
  $\Rightarrow$ Simplifies the analysis. Otherwise, study a M/G/1 queue
- Steady-state behavior (systems operating for a long time)
  $\Rightarrow$ Q: Limit probabilities for the M/M/1 system?

CTMC model

- Define CTMC by identifying states $Q(t)$ with queue lengths
  - $\Rightarrow$ Transition rates $q_{i,i+1} = \lambda$ for all i, and $q_{i,i-1} = \mu$ for $i \neq 0$
- Recall that first of two exponential times is exponentially distributed
  - $\Rightarrow$ Mean transition times are $\nu_i = \lambda + \mu$ for $i \neq 0$ and $\nu_0 = \lambda$



- Limit distribution equations (Rate out of j = Rate into j)

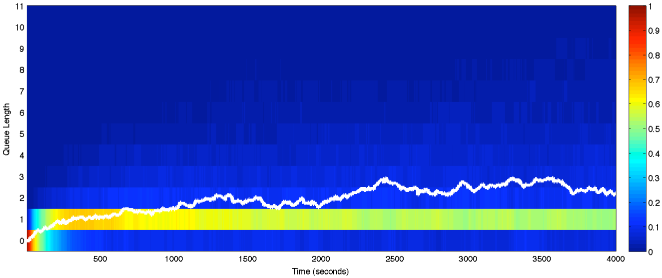
$$\lambda P_0 = \mu P_1, \qquad (\lambda + \mu) P_i = \lambda P_{i-1} + \mu P_{i+1}$$

Queue length as a function of time

- Simulation for $\lambda = 30$ customers/min, $\mu = 40$ services/min
- Probability distribution estimated by sample averaging with $M = 10^5$

$$\mathrm{P}\left(Q(t) = k\right) \approx \frac{1}{M} \sum_{i=1}^{M} \mathbb{I}\left\{Q_i(t) = k\right\}$$

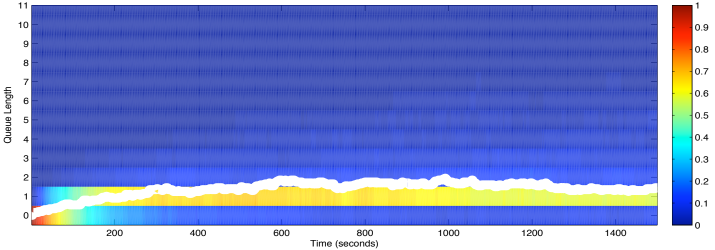
- Steady state (in a probabilistic sense) reached in around 10^3 mins.

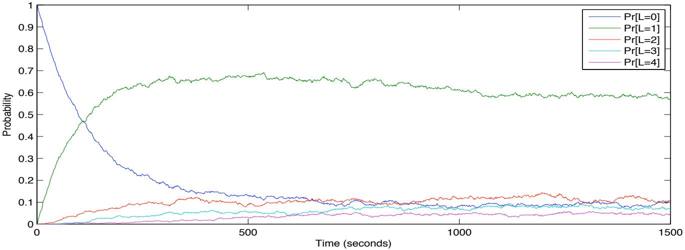


- Queue length vs. time. Probabilities are color coded
  - $\Rightarrow$ Mean queue length shown in white

Close up on initial times

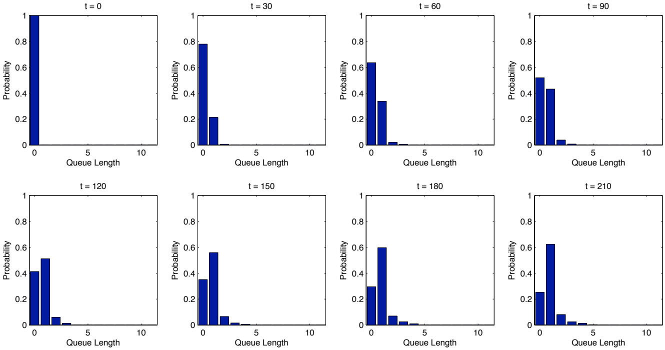
- Probabilities settle at their equilibrium values





Another view of queue length evolution

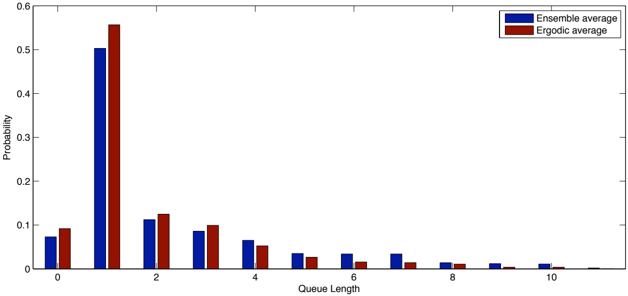
- Cross-sections of queue length probabilities at different times



Ergodicity

- Compare ensemble averages for large t with ergodic averages

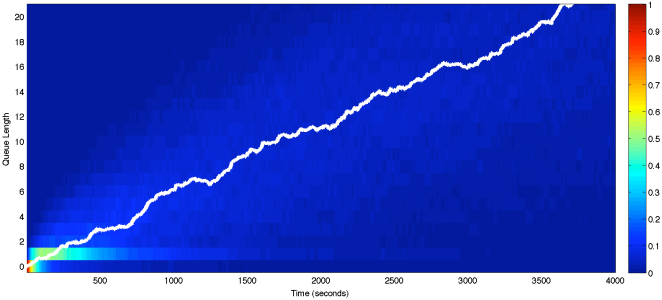
$$T_i(t) = \frac{1}{t}\int_0^t \mathbb{I}\left\{Q(\tau) = i\right\} d\tau$$



- They are approximately equal, as they should (equal as $t \to \infty$)

A non stable queue

- All former observations valid for stable queues ($\lambda < \mu$)
- Simulation for $\lambda = 60$ customers/min and $\mu = 40$, customers/min
  - $\Rightarrow$ Queue length grows unbounded
  - $\Rightarrow$ Probability of small number of customers in queue vanishes
  - $\Rightarrow$ Actually CTMC transient, $P_i \to 0$ for all i



- Queue length vs. time. Probabilities are color coded
  - $\Rightarrow$ Mean queue length shown in white

Solution of limit distribution equations

- Start expressing all prob. in terms of P_0. Definie traffic intensity $\rho := \lambda/\mu$
- Repeat process done for birth and death process
- Equation for P_0 $\Rightarrow$ $\Rightarrow \lambda P_0 = \mu P_1$
- Sum eqs. for P_1 and P_0 $\Rightarrow$

$$\begin{aligned} \lambda P_0 &= \mu P_1 \\ (\lambda+\mu)P_1 &= \lambda P_0 + \mu P_2 \end{aligned} \quad \Rightarrow \quad \lambda P_1 = \mu P_2$$

- Sum result and eq. for P_2 $\Rightarrow$

$$\begin{aligned} \lambda P_1 &= \mu P_2 \\ (\lambda+\mu)P_2 &= \lambda P_1 + \mu P_3 \end{aligned} \quad \Rightarrow \quad \lambda P_2 = \mu P_3$$

- Sum result and eq. for P_i $\Rightarrow$

$$\begin{aligned} \lambda P_{i-1} &= \mu P_i \\ (\lambda+\mu)P_i &= \lambda P_{i-1} + \mu P_{i+1} \end{aligned} \quad \Rightarrow \quad \lambda P_i = \mu P_{i+1}$$

- From where it follows $\Rightarrow P_{i+1} = (\lambda/\mu)P_i = \rho P_i$ and recursively $P_i = \rho^i P_0$

Solution of limit distribution equations (continued)

- The sum of all probabilities is 1 (use geometric series formula)

$$1 = \sum_{i=0}^{\infty} P_i = \sum_{i=0}^{\infty} \rho^i P_0 = \frac{P_0}{1-\rho}$$

- Solve for P_0 to obtain

$$P_0 = 1 - \rho, \qquad \Rightarrow \; P_i = (1-\rho)\rho^i$$

$\Rightarrow$ Valid for $\lambda/\mu < 1$, if not CTMC is transient (queue unstable)

- Expression coincides with non-concurrent queue in discrete time

  $\Rightarrow$ Not surprising. Continuous time $\approx$ discrete time with small Δt

  $\Rightarrow$ For small Δt non-concurrent hypothesis is accurate

- Present derivation "much cleaner," though

Steady-state expected queue length

- To compute expected queue length $\mathbb{E}\left[L\right]$ use limit probabilities

$$\mathbb{E}\left[L\right] = \sum_{i=0}^{\infty} iP_i = \sum_{i=0}^{\infty} i(1-\rho)\rho^i$$

- Latter is derivative of geometric sum $\left(\sum_{i=0}^{\infty} ix^i = x/(1-x)^2\right)$. Then

$$\mathbb{E}\left[L\right] = (1-\rho) \times \frac{\rho}{(1-\rho)^2} = \frac{\rho}{1-\rho}$$

- Recall $\lambda < \mu$ or equivalently $\rho < 1$ for queue stability

  $\Rightarrow$ If $\lambda \approx \mu$ queue is stable but $\mathbb{E}\left[L\right]$ becomes very large

Steady-state expected wait

- Customer arrives, L in queue already. Q: Time spent in queue?
  - $\Rightarrow$ Time required to service these L customers
  - $\Rightarrow$ Plus time until arriving customer is served
- Let $T_1, T_2, \ldots, T_{L+1}$ be these times. Queue wait $\Rightarrow W = \sum_{i=1}^{L+1} T_i$
- Expected value (condition on $L = \ell$, then expectation w.r.t. L)

$$\mathbb{E}\left[W\right] = \mathbb{E}\left[\sum_{i=1}^{L+1} T_i\right] = \mathbb{E}\left[\mathbb{E}\left[\sum_{i=1}^{\ell+1} T_i \,\middle|\, L = \ell\right]\right]$$

- $L = \ell$ "not random" in inner expectation $\Rightarrow$ interchange with sum

$$\mathbb{E}\left[W\right] = \mathbb{E}\left[\sum_{i=1}^{L+1} \mathbb{E}\left[T_i\right]\right] = \mathbb{E}\left[(L+1)\mathbb{E}\left[T_i\right]\right] = \mathbb{E}\left[L+1\right]\mathbb{E}\left[T_i\right]$$

Expected wait (continued)

- Use expression for $\mathbb{E}[L]$ to evaluate $\mathbb{E}[L+1]$ as

$$\mathbb{E}[L+1] = \mathbb{E}[L] + 1 = \frac{\rho}{1-\rho} + 1 = \frac{1}{1-\rho}$$

- Substitute expressions for $\mathbb{E}[L+1]$ and $\mathbb{E}[T_i] = 1/\mu$

$$\mathbb{E}[W] = \frac{1}{\mu} \times \frac{1}{1-\rho} = \frac{1}{\mu - \lambda}$$

- Recall $\lambda =$ arrival rate. Former may be written as

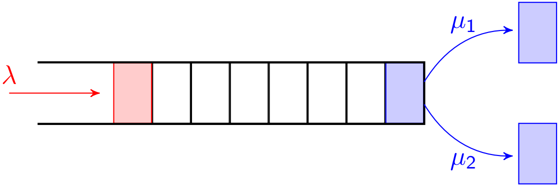
$$\mathbb{E}[W] = \frac{1}{\lambda} \times \frac{\rho}{1-\rho} = (1/\lambda)\mathbb{E}[L]$$

Little's law

- For M/M/1 queue have just seen $\Rightarrow \mathbb{E}[L] = \lambda \mathbb{E}[W]$
  - $\Rightarrow$ Expression referred to as Little's law
- True even if arrivals and departures are not Poisson (not proved)
- Expected nr.customers in queue = arrival rate $\times$ expected wait

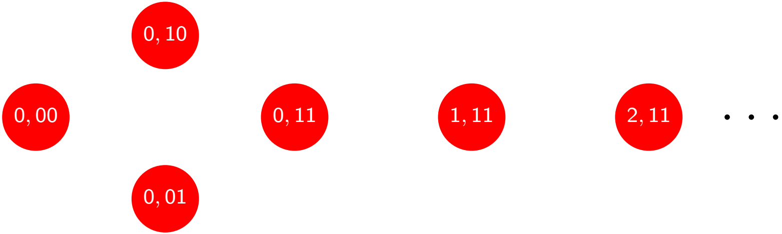
M/M/2 queue

- Service offered by two Poisson servers with service rates μ_1 and μ_2
  - $\Rightarrow$ Arrivals are Poisson with rate λ as in the M/M/1 queue
- When a server finishes serving a customer, serves next one in queue
  - $\Rightarrow$ If queue is empty the server waits for the next customer
- If both servers are idle when a new customer arrives
  - $\Rightarrow$ Service is performed by server 1 (simply by convention)



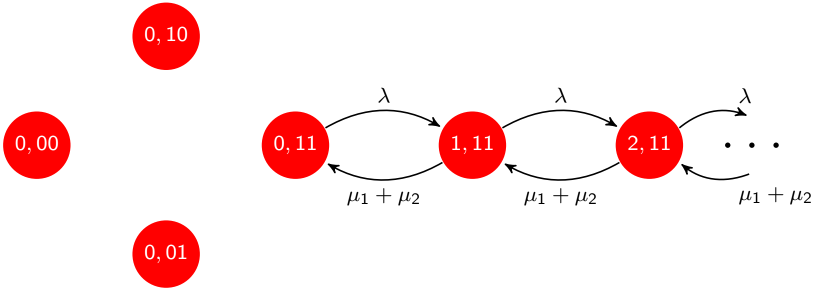
CTMC model: States

- When no customers are in line, need to distinguish servers' states
  - State $0, 00$ = no customers in queue, no customers being served
  - State $0, 10$ = no customers in queue, 1 customer served by server 1
  - State $0, 01$ = no customers in queue, 1 customer served by server 2
  - State $0, 11$ = no customers in queue, 2 customers in service
- When there are customers in line, both servers are busy
  - State $i, 11$ = $i > 0$ customers in queue and 2 customers in service
  - States $i, 01$, $i, 10$ and $i, 00$ are not possible for $i > 0$



CTMC model: Transition rates

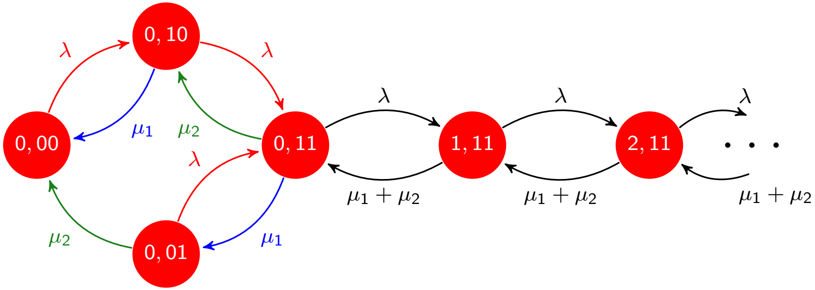
- Transition from $i, 11$ to $(i+1, 11)$ when arrival $\Rightarrow q_{i,11;(i+1),11} = \lambda$
- Transition from $i, 11$ to $(i-1, 11)$ when either server 1 or 2 finishes
  $\Rightarrow$ First service completion by either server 1 or 2
- Min. of two exponentials is exponential $\Rightarrow q_{i,11;(i-1),11} = \mu_1 + \mu_2$



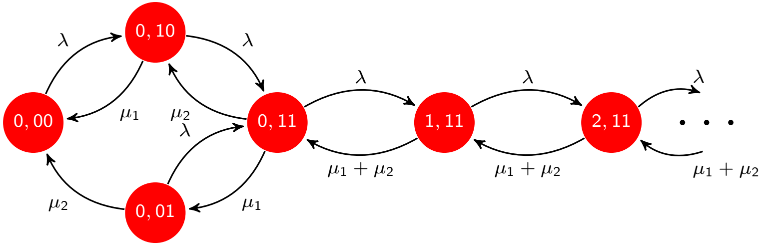
CTMC model: Transition rates (continued)

- From $0,00$ move to $0,10$ on arrival $\Rightarrow q_{0,00;0,10} = \lambda$
- From $0,10$ move to $0,11$ on arrival $\Rightarrow q_{0,10;0,11} = \lambda$
- From $0,01$ move to $0,11$ on arrival $\Rightarrow q_{0,01;0,11} = \lambda$

- From $0,10$ to $0,00$ when server 1 finishes $\Rightarrow q_{0,01;0,00} = \mu_1$
- From $0,11$ to $0,01$ when server 1 finishes $\Rightarrow q_{0,11;0,01} = \mu_1$
- From $0,01$ to $0,00$ when server 2 finishes $\Rightarrow q_{0,01;0,00} = \mu_2$
- From $0,11$ to $0,10$ when server 2 finishes $\Rightarrow q_{0,11;0,10} = \mu_2$



Limit distribution equations



- For states $i, 11$ with $i > 0$, eqs. are analogous to M/M/1 queue

$$(\lambda + \mu_1 + \mu_2)P_{i,11} = \lambda P_{(i-1),11} + (\mu_1 + \mu_2)P_{(i+1),11}$$

- For states $0, 11$, $0, 10$, $0, 01$ and $0, 00$ we have

$$\begin{aligned}
(\lambda + \mu_1 + \mu_2)\, P_{0,11} &= \lambda P_{0,10} \;+ \lambda P_{0,01} \;+ (\mu_1 + \mu_2)P_{1,11} \\
(\lambda + \mu_1)\, P_{0,10} &= \lambda P_{0,00} \;+ \mu_2 P_{0,11} \\
(\lambda + \mu_2)\, P_{0,01} &= \mu_1 P_{0,11} \\
\lambda\, P_{0,00} &= \mu_1 P_{0,10} + \mu_2 P_{0,01}
\end{aligned}$$

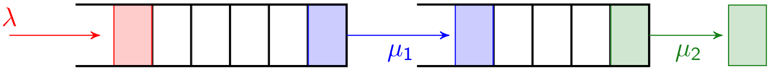
- System of linear equations $\Rightarrow$ Solve numerically to find probabilities

Closing comments

- For large i behaves like M/M/1 queue with service rate $(\mu_1 + \mu_2)$
  $\Rightarrow$ Still, states with no queued packets are important
- M/M/c queue $\Rightarrow$ c servers with rates $\mu_1, \ldots, \mu_c$
  $\Rightarrow$ More cumbersome to analyze but no fundamental differences

A queue tandem

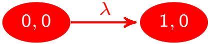
- Customers arrive at system to receive two services
- They arrive at a rate λ and wait in queue 1 for service 1
  $\Rightarrow$ Service 1 is performed at a rate μ_1
- After completions of service 1 customers move to queue 2
  $\Rightarrow$ Service 2 is performed at a rate μ_2



CTMC model

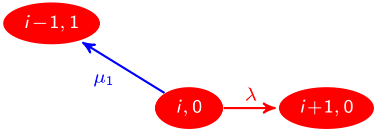
- States (i, j) represent i customers in queue 1 and j in queue 2
- If both queues are empty ($i = j = 0$), only possible event is an arrival

$$q_{00,10} = \lambda$$



- If queue 2 is empty might have arrival or completion of service 1

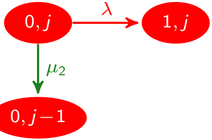
$$q_{i0,(i+1)0} = \lambda$$
$$q_{i0,(i-1)1} = \mu_1$$



CTMC model (continued)

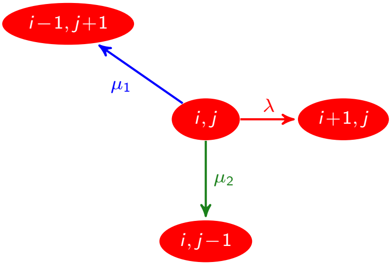
- If queue 1 is empty might have arrival or completion of service 2

$$q_{0j,1j} = \lambda$$
$$q_{0j,0(j-1)} = \mu_2$$



- If no queue is empty arrival, service 1 and service 2 possible

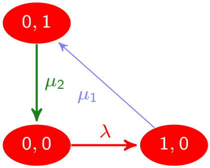
$$q_{ij,(i+1)j} = \lambda$$
$$q_{ij,(i-1)(j+1)} = \mu_1$$
$$q_{ij,i(j-1)} = \mu_2$$



Balance equations

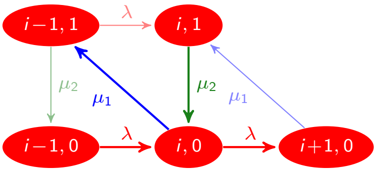
- Rate at which CTMC enters state (i, j) = rate at which CTMC leaves (i, j)
- **State** $(\mathbf{0}, \mathbf{0})$ - Both queues empty
- From $(0, 0)$ can go to $(1, 0)$
- Can enter $(0, 0)$ from $(0, 1)$

$$\lambda P_{00} = \mu_2 P_{01}$$



- **State** $(\mathbf{i}, \mathbf{0})$ - Queue 2 empty
- From $(i, 0)$ go to $(i + 1, 0)$ or $(i - 1, 1)$
- Into $(i, 0)$ from $(i - 1, 0)$ or $(i, 1)$

$$(\lambda + \mu_1)P_{i0} = \lambda P_{(i-1)0} + \mu_2 P_{i1}$$

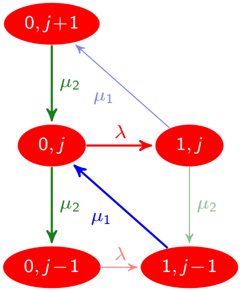


Balance equations (continued)

- **State $(\mathbf{0}, \mathbf{j})$** - Queue 1 empty

- From $(0, j)$ go to $(1, j)$ or $(0, j-1)$
- Into $(0, j)$ from $(1, j-1)$ or $(0, j+1)$

$$(\lambda + \mu_2)P_{0j} = \mu_1 P_{1(j-1)} + \mu_2 P_{0(j+1)}$$

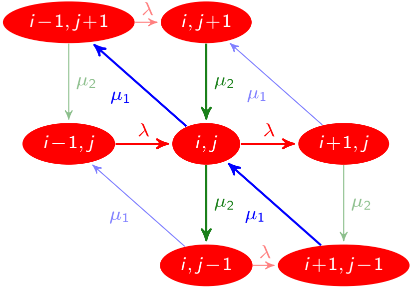


Balance equations (continued)

- **State (i, j)** - Neither queue empty

- From (i,j) can go to $(i+1,j)$, $(i-1,j+1)$ or $(i,j-1)$
- Can enter (i,j) from $(i-1,j)$, $(i+1,j-1)$ or $(i,j+1)$

$$(\lambda + \mu_1 + \mu_2)P_{ij} = \lambda P_{(i-1)j} + \mu_1 P_{(i+1)(j-1)} + \mu_2 P_{i(j+1)}$$



Solution of balance equations

- Direct substitution shows that balance equations are solved by

$$P_{ij} = \left(1 - \frac{\lambda}{\mu_1}\right)\left(\frac{\lambda}{\mu_1}\right)^i \left(1 - \frac{\lambda}{\mu_2}\right)\left(\frac{\lambda}{\mu_2}\right)^j$$

- Compare with expression for M/M/1 queue

  $\Rightarrow$ It behaves as two independent M/M/1 queues

  $\Rightarrow$ First queue has rates λ and μ_1

  $\Rightarrow$ Second queue has rates λ and μ_2

- Result can be generalized to networks of queues

  $\Rightarrow$ Important in transportation networks

  $\Rightarrow$ Also useful to analyze Internet traffic