CS4277 / CS5477
3D Computer Vision

Lecture 2:
3D projective geometry, Circular points and Absolute conic

Assoc. Prof. Lee Gim Hee

AY 2022/23

Semester 2

Points in $\mathbb{P}^3$

- A point $\mathbf{X}$ in 3-space is represented in homogeneous coordinates as a 4-vector, i.e.

$$\mathbf{X} = (X_1, X_2, X_3, X_4)^\top \text{ with } \quad X_4 \neq 0$$

- Represents the point $(X, Y, Z)^\top$ of $\mathbb{R}^3$ with inhomogeneous coordinates

$$X = X_1/X_4, \; Y = X_2/X_4, \; Z = X_3/X_4.$$

- Homogeneous points with $X_4 = 0$ represent points at infinity.

Projective Transformation of Points in $\mathbb{P}^3$

- A projective transformation acting on $\mathbb{P}^3$ is a linear transformation on $\mathbf{X}$ by a non-singular 4×4 matrix:

$$\mathbf{X}' = \mathrm{H}\mathbf{X}.$$

- The matrix H is homogeneous and has 15 degrees of freedom: 16 elements less one for scaling.
- As in $\mathbb{P}^2$, the map is a collineation (lines are mapped to lines),
- which preserves incidence relations such as the intersection point of a line with a plane, and order of contact.

Planes in $\mathbb{P}^3$

- A plane in 3-space may be written as:

$$\pi_1 \mathrm{X} + \pi_2 \mathrm{Y} + \pi_3 \mathrm{Z} + \pi_4 = 0.$$

- Homogenizing by $\mathrm{X} \mapsto \mathrm{X}_1/\mathrm{X}_4, \mathrm{Y} \mapsto \mathrm{X}_2/\mathrm{X}_4, \mathrm{Z} \mapsto \mathrm{X}_3/\mathrm{X}_4$ gives

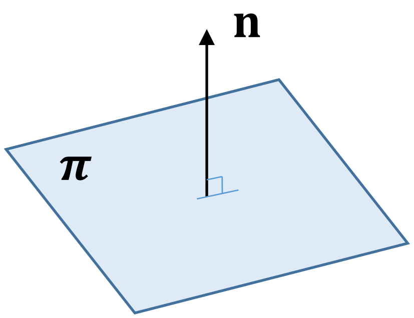
$$\pi_1 \mathrm{X}_1 + \pi_2 \mathrm{X}_2 + \pi_3 \mathrm{X}_3 + \pi_4 \mathrm{X}_4 = 0 \quad \text{or} \quad \boldsymbol{\pi}^\mathsf{T} \mathbf{X} = 0,$$

which expresses that the point $\mathbf{X}$ is on the plane $\boldsymbol{\pi} = (\pi_1, \pi_2, \pi_3, \pi_4)^\mathsf{T}$.

Planes in $\mathbb{P}^3$

- Only three independent ratios $\{\pi_1 : \pi_2 : \pi_3 : \pi_4\}$ of the plane coefficients are significant, i.e., 3 degrees of freedom.

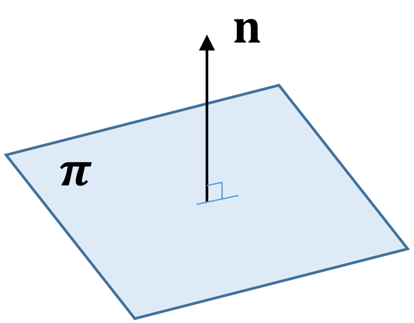
- The first 3 components of $\boldsymbol{\pi}$ correspond to the plane normal of Euclidean geometry, i.e., $\mathbf{n} = (\pi_1, \pi_2, \pi_3)^\top$.



Planes in $\mathbb{P}^3$

- Using inhomogenous notation to rewrite $\boldsymbol{\pi}^\mathsf{T}\mathrm{X} = 0$ as:

$$\mathbf{n}.\widetilde{\mathbf{X}} + d = 0,$$



where $\mathrm{X} = (\mathrm{X}, \mathrm{Y}, \mathrm{Z}, 1)^\mathsf{T}$ and $d = \pi_4$.

- In this form, $d/||\mathbf{n}||$ is the distance of the plane from the origin.

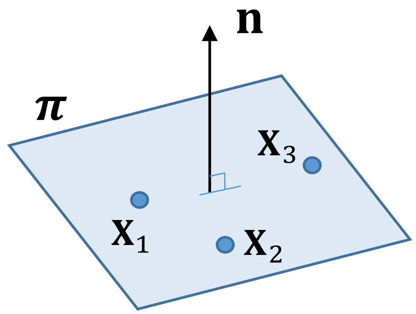
- Under the point transformation $\mathbf{X}' = \mathrm{H}\mathbf{X}$, a plane transforms as:

$$\boldsymbol{\pi}' = \mathrm{H}^{-\mathsf{T}}\boldsymbol{\pi}.$$

Three Points Define a Plane

- Suppose three points $\mathbf{X}_i$ are incident with the plane $\boldsymbol{\pi}$, where each point satisfies $\boldsymbol{\pi}^\top \mathbf{X}_i = 0$ for $i = 1,2,3$, i.e.

$$\begin{bmatrix} \mathbf{X}_1^\top \\ \mathbf{X}_2^\top \\ \mathbf{X}_3^\top \end{bmatrix} \boldsymbol{\pi} = \mathbf{0}.$$



- The 3×4 matrix $[\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3]^\top$ has rank 3 when the points are in general positions, i.e., linearly independent.

- The plane $\boldsymbol{\pi}$ defined by the points is obtained uniquely (up to scale) as the 1-dimensional (right) null-space.

Three Points Define a Plane

- If the matrix $[\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_2]^\top$ has only a rank of 2, and consequently the null-space is 2-dimensional.

- Then the points are collinear and define a pencil of planes with the line of collinear points as axis.

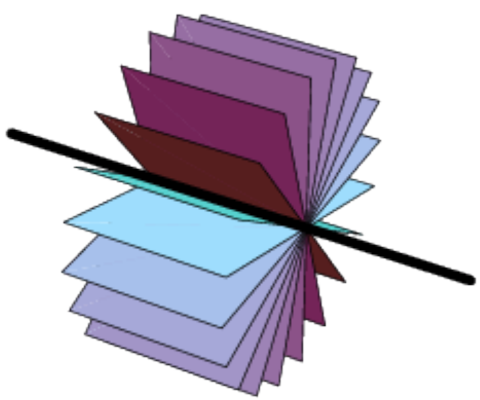
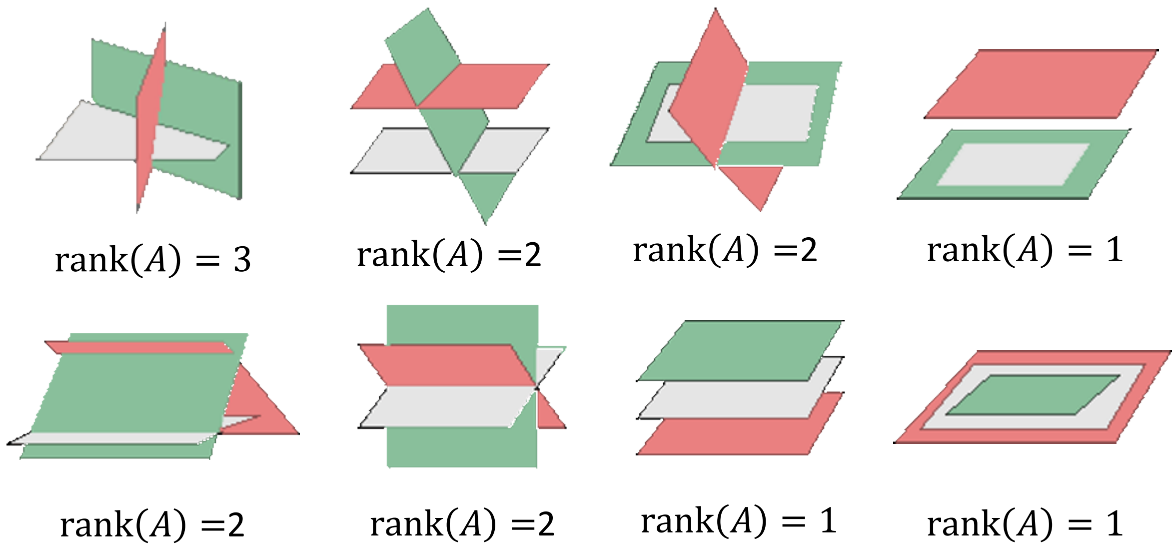


Image source: https://en.wikipedia.org/wiki/Sheaf_of_planes

Three Planes Define a Point

- The intersection point $\mathbf{X}$ of three planes $\boldsymbol{\pi}_i$ can be computed as the (right) null-space of the 3×4 matrix composed of the planes as rows:

$$\underbrace{\begin{bmatrix} \boldsymbol{\pi}_1^{\top} \\ \boldsymbol{\pi}_2^{\top} \\ \boldsymbol{\pi}_3^{\top} \end{bmatrix}}_{A} \mathbf{X} = \mathbf{0}.$$



- The development here is dual to the case of three points defining a plane and it shows the point-plane duality.

Image source: https://www.ditutor.com/space/three_planes.html
Refer to link for details of the eight possibilities.

Parametrized Points on a Plane

- The points $\mathbf{X}$ on the plane $\boldsymbol{\pi}$ may be written as

$$\mathbf{X} = \mathtt{M}\mathbf{x} .$$

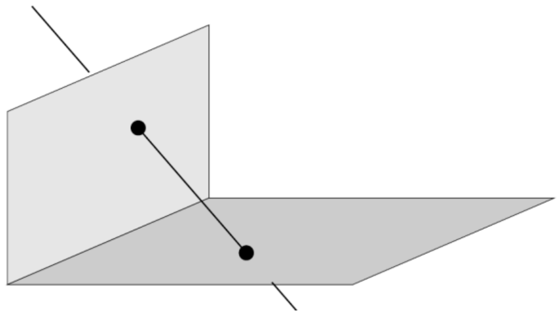
- The columns of the 4×3 matrix M generate the rank 3 null-space of $\boldsymbol{\pi}^\top$, i.e., $\boldsymbol{\pi}^\top \mathrm{M} = \mathbf{0}_{1\times 3}$, and the 3-vector $\mathbf{x}$ parametrizes points on the plane $\boldsymbol{\pi}$.

- M is not unique, suppose the plane is $\boldsymbol{\pi} = (a, b, c, d)^\top$ and a is non-zero, then $\mathrm{M}^\top$ can be written as

$$\mathrm{M}^\top = [\mathbf{p} \mid I_{3\times 3}], \qquad \text{where } \mathbf{p} = \left(-\frac{b}{a}, -\frac{c}{a}, -\frac{d}{a}\right)^\top .$$

Lines in $\mathbb{P}^3$

- A line is defined by the join of two points or the intersection of two planes.



- Lines have 4 degrees of freedom in 3-space.

Sketch of Proof: A line may be specified by its points of intersection with two orthogonal planes. Each intersection point has 2 degrees of freedom, hence 4 degrees of freedom.

- Awkward to represent 3-space line with a homogeneous 5-vector, we will look at two alternatives representations.

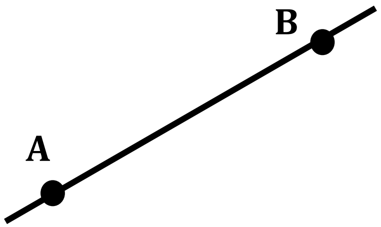
Image source: "Multiple View Geometry in Computer Vision", Richard Hartley and Andrew Zisserman

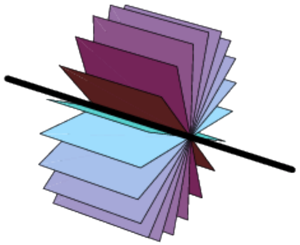
Lines in $\mathbb{P}^3$: Null-Space and Span Representation

- Suppose $\mathbf{A}, \mathbf{B}$ are two (non-coincident) space points.

- The line joining these points (6 dofs, i.e. overparameterized) is represented by the span of the row space of the 2×4 matrix W composed of $\mathbf{A}^\mathsf{T}$ and $\mathbf{B}^\mathsf{T}$ as rows:

$$\mathtt{W} = \begin{bmatrix} \mathbf{A}^\mathsf{T} \\ \mathbf{B}^\mathsf{T} \end{bmatrix}.$$





- Then:

1. The span of $\mathtt{W}^\mathsf{T}$ is the pencil of points $\lambda\mathbf{A} + \mu\mathbf{B}$ on the line.

2. The span of the 2-dimensional right null-space of W is the pencil of planes with the line as axis.

Image source: https://en.wikipedia.org/wiki/Sheaf_of_planes

Lines in $\mathbb{P}^3$: Null-Space and Span Representation

Remarks on (1):

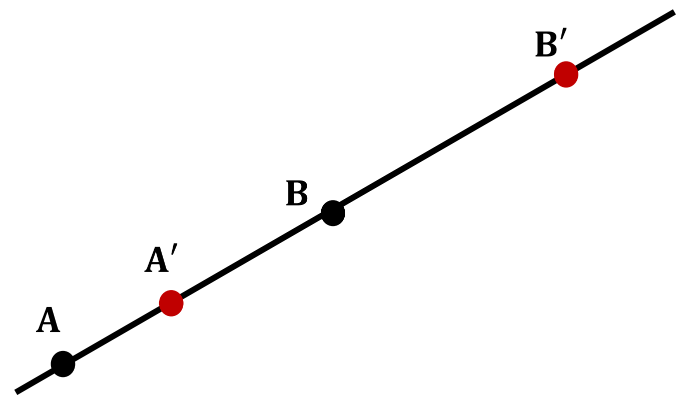
- It is evident that two other points, $\mathbf{A}'^{\mathsf{T}}$ and $\mathbf{B}'^{\mathsf{T}}$, on the line will generate a matrix W' with the same span as W.

- Hence, the representation is independent of the particular points used to define it.

Same line!

$$\mathrm{W} = \begin{bmatrix} \mathrm{A}^{\mathsf{T}} \\ \mathrm{B}^{\mathsf{T}} \end{bmatrix}$$

$$\mathrm{W}' = \begin{bmatrix} \mathrm{A}'^{\mathsf{T}} \\ \mathrm{B}'^{\mathsf{T}} \end{bmatrix}$$



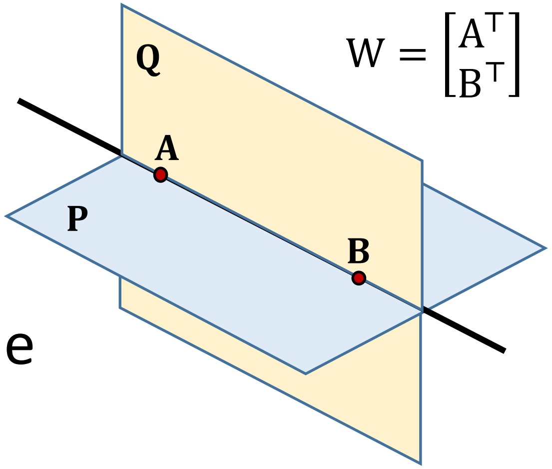
Lines in $\mathbb{P}^3$: Null-Space and Span Representation

Remarks on (2):

- Suppose that $\mathbf{P}$ and $\mathbf{Q}$ are a basis for the null-space, then $\mathrm{W}\mathbf{P} = \mathbf{0}$ and consequently $\mathbf{A}^\top\mathbf{P} = \mathbf{B}^\top\mathbf{P} = 0$, so that $\mathbf{P}$ is a plane containing the points $\mathbf{A}$ and $\mathbf{B}$.

- Similarly, $\mathbf{Q}$ is a distinct plane also containing the points $\mathbf{A}$ and $\mathbf{B}$.

- $\mathbf{A}$ and $\mathbf{B}$ lie on both the (linearly independent) planes $\mathbf{P}$ and $\mathbf{Q}$, so the line defined by W is the plane intersection.



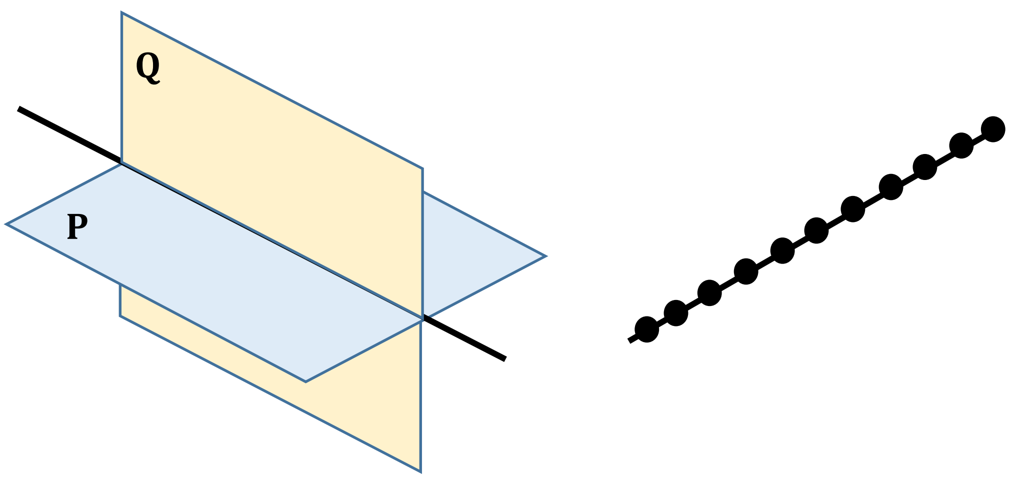
- Any plane of the pencil, with the line as axis, is given by the span $\lambda\mathbf{P} + \mu\mathbf{Q}$.

Lines in $\mathbb{P}^3$: Null-Space and Span Representation

- The dual representation of a line as the intersection of two planes, $\mathbf{P}, \mathbf{Q}$, follows in a similar manner.

- The line is represented as the span (of the row space) of the 2×4 matrix $\mathtt{W}^*$ composed of $\mathbf{P}^\top$ and $\mathbf{Q}^\top$ as rows:

$$\mathtt{W}^* = \begin{bmatrix} \mathbf{P}^\top \\ \mathbf{Q}^\top \end{bmatrix}$$



Lines in $\mathbb{P}^3$: Null-Space and Span Representation

- With the properties:

1. The span of $\mathrm{W}^{*\top}$ is the pencil of planes $\lambda\mathbf{P} + \mu\mathbf{Q}$ with the line as axis.

2. The span of the 2-dimensional null-space of W^* is the pencil of points on the line.

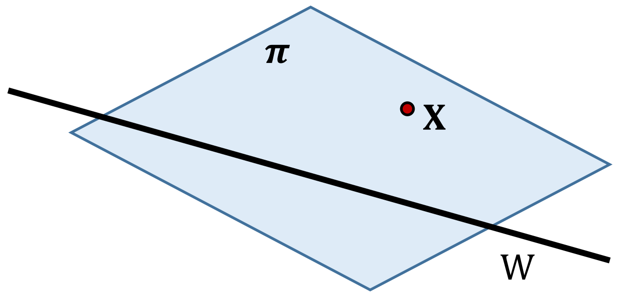
- The two representations are related by $\mathrm{W}^*\mathrm{W}^\top = \mathrm{W}\mathrm{W}^{*\top} = 0_{2\times 2}$, where $0_{2\times 2}$ is a 2×2 null matrix.

Lines in $\mathbb{P}^3$: Null-Space and Span Representation

- Join and incidence relations are also computed from null-spaces:

1. The plane $\boldsymbol{\pi}$ defined by the join of the point $\mathbf{X}$ and line W is obtained from the null-space of

$$\mathtt{M} = \begin{bmatrix} \mathtt{W} \\ \mathbf{X}^\mathsf{T} \end{bmatrix}.$$



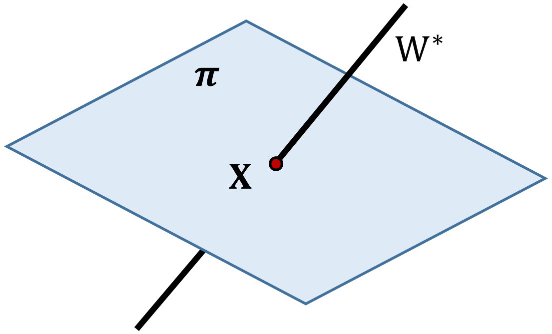
If the null-space of M is 2-dimensional then $\mathbf{X}$ is on W, otherwise $\mathtt{M}\boldsymbol{\pi} = \mathbf{0}$.

Lines in $\mathbb{P}^3$: Null-Space and Span Representation

- Join and incidence relations are also computed from null-spaces:

2. The point $\mathbf{X}$ defined by the intersection of the line W with the plane $\boldsymbol{\pi}$ is obtained from the null-space of

$$\mathtt{M} = \begin{bmatrix} \mathtt{W}^* \\ \boldsymbol{\pi}^\mathsf{T} \end{bmatrix}.$$



If the null-space of M is 2-dimensional then the line W is on $\boldsymbol{\pi}$, otherwise $\mathrm{M}\mathbf{X} = 0$.

3D Hierarchy of Transformations

$$\mathrm{R} = \begin{bmatrix} c_1 c_2 & c_1 s_2 s_3 - c_3 s_1 & s_1 s_3 + c_1 c_3 s_2 \\ c_2 s_1 & c_1 c_3 + s_1 s_2 s_3 & c_3 s_1 s_2 - c_1 s_3 \\ -s_2 & c_2 s_3 & c_2 c_3 \end{bmatrix},$$ 3x3 rotation matrix

$t = (t_x, t_y, t_z)^\top$, 3x1 translation vector

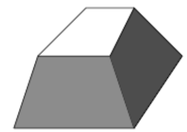
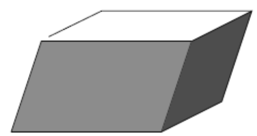
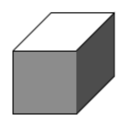
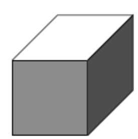
Group	Matrix	Distortion	Invariant properties
Projective 15 dof	$\begin{bmatrix} \mathrm{A} & \mathbf{t} \\ \mathbf{v}^\top & v \end{bmatrix}$		Intersection and tangency of surfaces in contact.
Affine 12 dof	$\begin{bmatrix} \mathrm{A} & \mathbf{t} \\ \mathbf{0}^\top & 1 \end{bmatrix}$		Parallelism of planes, volume ratios, centroids. The plane at infinity, π_∞,
Similarity 7 dof	$\begin{bmatrix} s\mathrm{R} & \mathbf{t} \\ \mathbf{0}^\top & 1 \end{bmatrix}$		The absolute conic, Ω_∞,
Euclidean 6 dof	$\begin{bmatrix} \mathrm{R} & \mathbf{t} \\ \mathbf{0}^\top & 1 \end{bmatrix}$		Volume.

Table source: "Multiple View Geometry in Computer Vision", Richard Hartley and Andrew Zisserman

Line at Infinity and Circular Points

- In the following, it will be shown that:

1. The projective distortion may be removed once the image of $\mathbf{l}_\infty$ is specified;
2. And the affine distortion removed once the image of the circular points is specified.

- Then the only remaining distortion is a similarity.

The Line at Infinity

- The line at infinity, $\mathbf{l}_\infty$, is a fixed line under the projective transformation H if and only if H is an affinity, i.e.,

$$\mathbf{l}'_\infty = \mathrm{H}_\mathrm{A}^{-\mathsf{T}}\mathbf{l}_\infty = \begin{bmatrix} \mathrm{A}^{-\mathsf{T}} & \mathbf{0} \\ -\mathbf{t}^\mathsf{T}\mathrm{A}^{-\mathsf{T}} & 1 \end{bmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \mathbf{l}_\infty .$$

- An affinity is the most general linear transformation with $\mathrm{H}_{31} = \mathrm{H}_{32} = 0$ for the relationship to be true.

- We will see that identifying $\mathbf{l}_\infty$ allows the recovery of affine properties (parallelism, ratio of lengths).

The Line at Infinity

- Contrast this with projective transformation, where an ideal point and line at infinity might not remain at infinity.

$$\mathrm{H}_p\mathbf{x} = \mathbf{x}' \quad \Rightarrow \quad \begin{bmatrix} \mathtt{A} & \mathbf{t} \\ \mathbf{v}^\mathsf{T} & v \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} = \begin{pmatrix} \mathtt{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ \boxed{v_1 x_1 + v_2 x_2} \end{pmatrix}$$

Might not be 0 since v_1 and v_2 are not 0.

$$\mathrm{H}_p^{-\mathsf{T}}\mathbf{l} = \mathbf{l}' \quad \Rightarrow \quad \begin{bmatrix} \mathrm{A} & \mathbf{t} \\ \mathbf{v}^\mathsf{T} & v \end{bmatrix}^{-\mathsf{T}} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \boxed{\begin{matrix} a_{21}v_2 - a_{22}v_1 \\ -a_{11}v_2 + a_{12}v_1 \end{matrix}} \\ a_{11}a_{22} - a_{12}a_{21} \end{pmatrix}$$

Might not be 0 since v_1 and v_2 are not 0.

The Line at Infinity

- Interestingly, $\mathbf{l}_\infty$ is not fixed pointwise under an affine transformation.

- In general, under an affinity , a point on $\mathbf{l}_\infty$ (an ideal point) is mapped to another point on $\mathbf{l}_\infty$:

$$\begin{bmatrix} \mathtt{A} & \mathbf{t} \\ \mathbf{0}^\mathsf{T} & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} = \begin{pmatrix} \mathtt{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ 0 \end{pmatrix}.$$

- Nonetheless, it would be the same point when:

$$\mathtt{A}(x_1, x_2)^\mathsf{T} = k(x_1, x_2)^\mathsf{T}.$$

Recovery of Affine Properties from Images

Affine Rectification: imaged line at infinity can be used to remove projective distortion.

H_p maps ideal points and l_∞ to finite

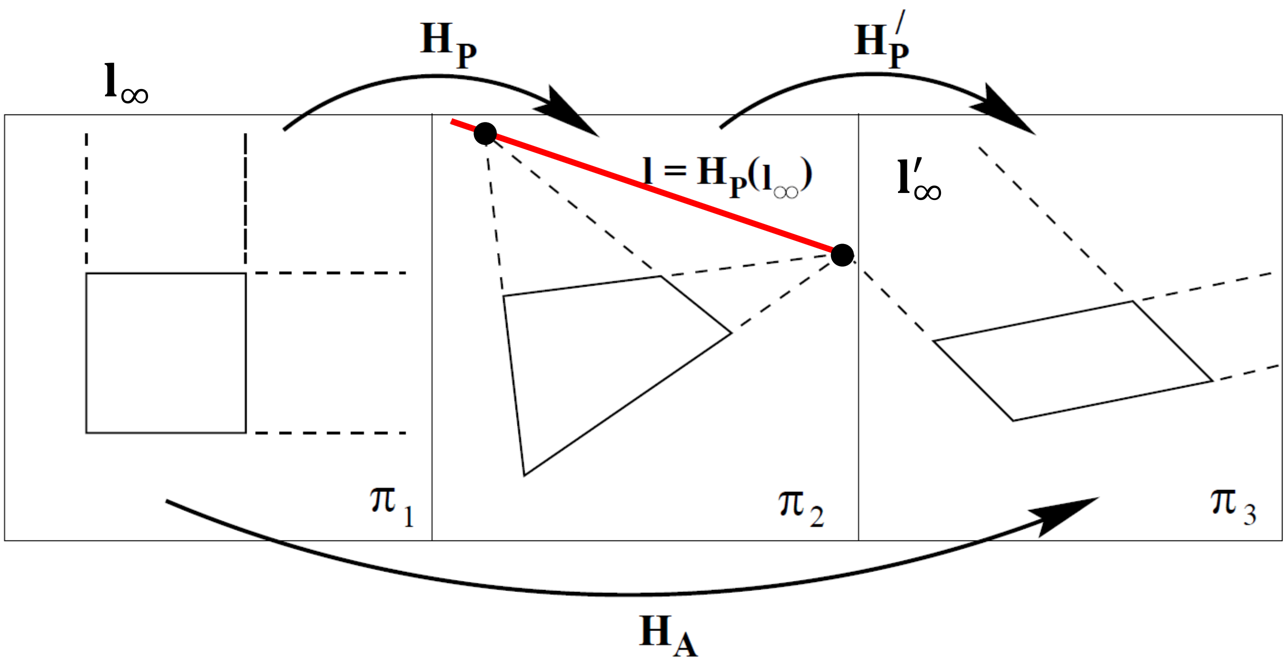


Image source: "Multiple View Geometry in Computer Vision", Richard Hartley and Andrew Zisserman

Recovery of Affine Properties from Images

Problem:

Given $\mathbf{l} = (l_1, l_2, l_3)^\top$ where $l_3 \neq 0$**,** find H_p' that can be used to remove the projective distortion.

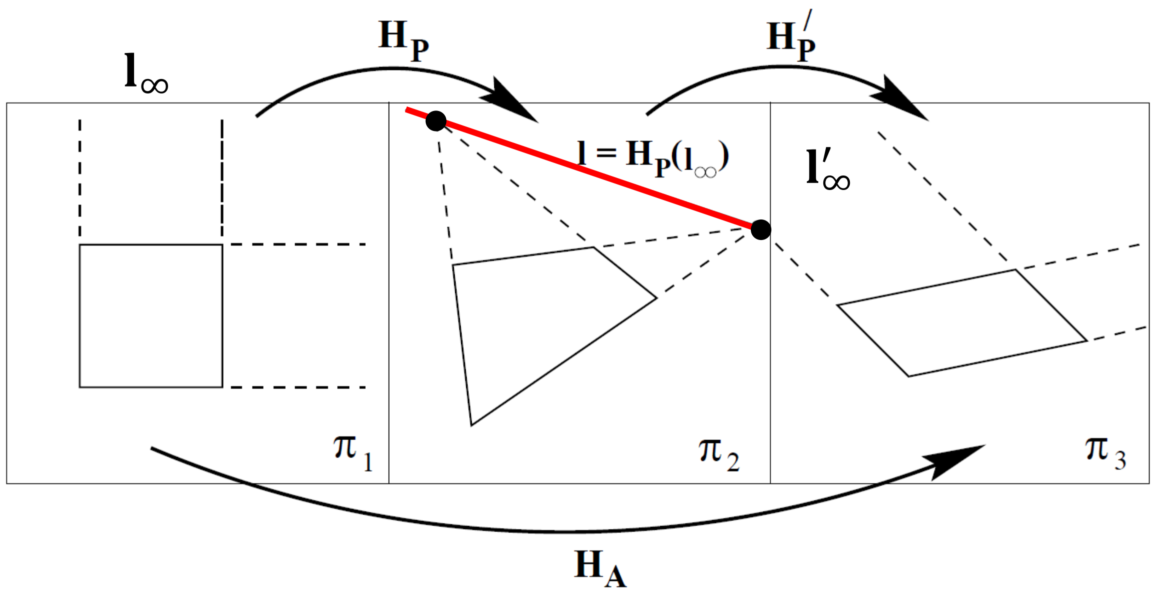


Image source: “Multiple View Geometry in Computer Vision”, Richard Hartley and Andrew Zisserman

Recovery of Affine Properties from Images

Solution:

Since $\mathbf{l} = \mathrm{H}_p^{-\top}\mathbf{l}_\infty \Rightarrow \mathrm{H}_\mathrm{p}^\top [l_1, l_2, l_3]^\top = [0,0,1]^\top$, we can choose

$$\mathrm{H}_p^\top = \begin{pmatrix} 1 & 0 & -l_1/l_3 \\ 0 & 1 & -l_2/l_3 \\ 0 & 0 & 1/l_3 \end{pmatrix}.$$

Furthermore, $\mathrm{H}_A = \mathrm{H}'_p \mathrm{H}_P$

$$\Rightarrow \mathrm{H}'_p = \mathrm{H}_A \mathrm{H}_p^{-1} = \mathrm{H}_A \begin{pmatrix} 1 & 0 & -l_1/l_3 \\ 0 & 1 & -l_2/l_3 \\ 0 & 0 & 1/l_3 \end{pmatrix}^{-\top},$$

where H_A is any affine transformation since $\mathbf{l}'_\infty = \mathrm{H}_A^{-\top}\mathbf{l}_\infty$.

Recovery of Affine Properties from Images

1. The imaged vanishing line of the plane $\mathbf{l}$ is computed from the intersection of two sets of imaged parallel lines.
2. Compute $\mathrm{H}_p' = \mathrm{H}_A \mathrm{H}_P^{-1}$ by choosing an arbitrary affinity H_A.

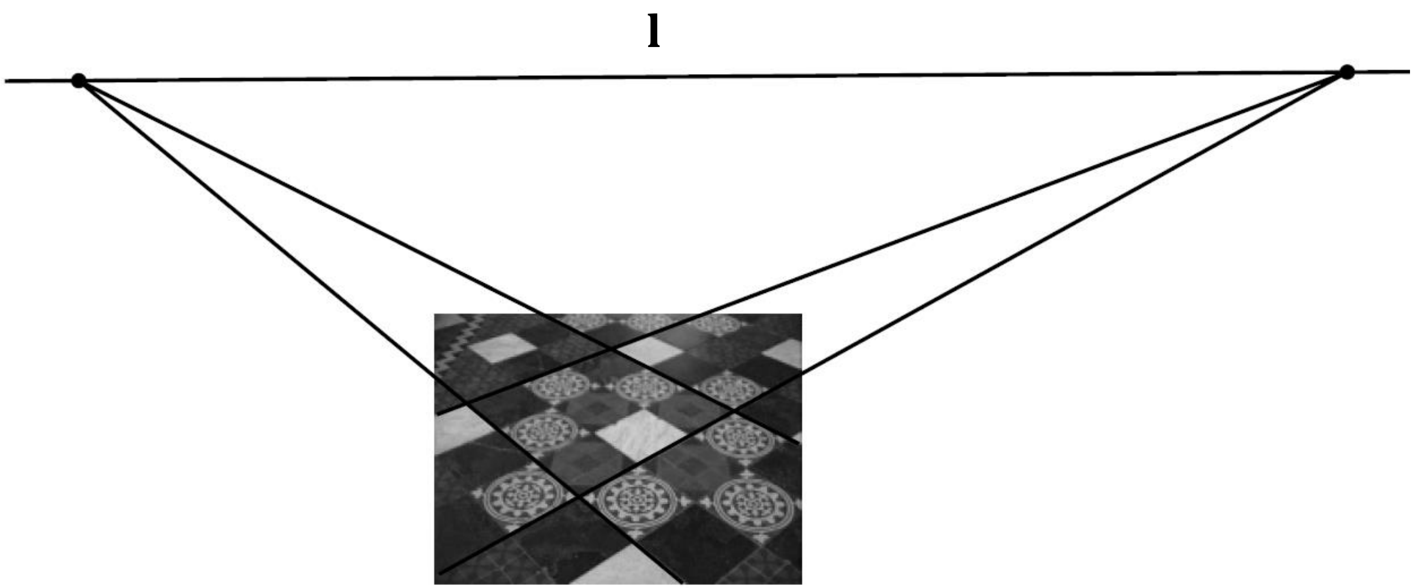


Image source: "Multiple View Geometry in Computer Vision", Richard Hartley and Andrew Zisserman

Recovery of Affine Properties from Images

3. Use H'_p to projectively warp the image to produce the affinely rectified image.

4. Affine properties can be recovered from the affinely rectified image, e.g. parallel lines and ratio of lengths.

5. Note: angles cannot be recovered since image is still affinely distorted.

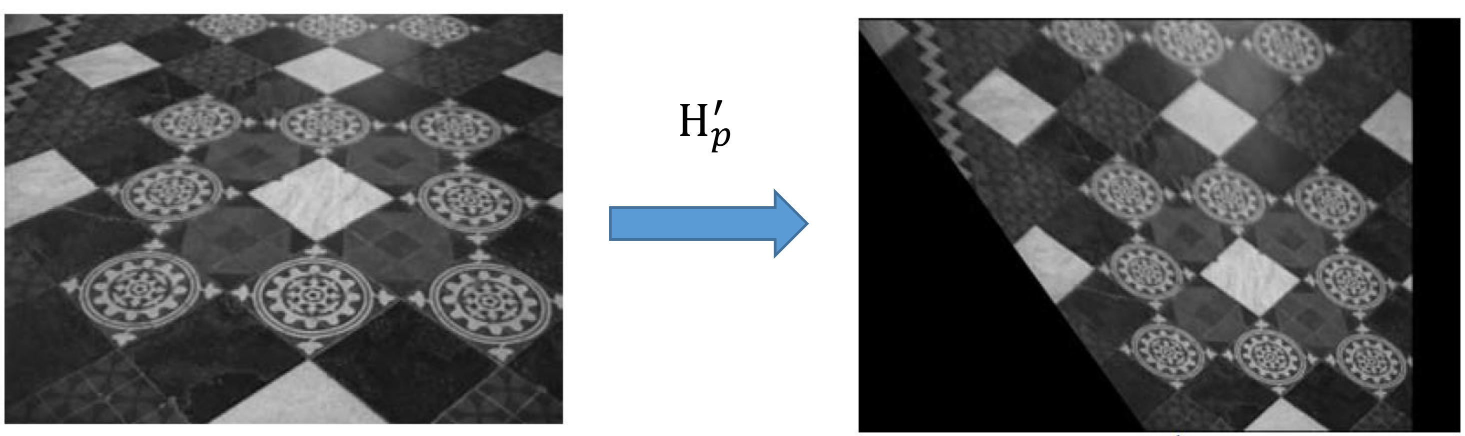
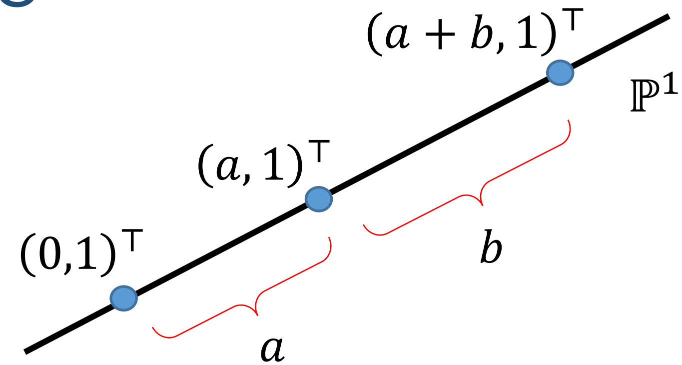


Image source: “Multiple View Geometry in Computer Vision”, Richard Hartley and Andrew Zisserman

Computing a Vanishing Point from a Length Ratio

- Conversely, known affine properties may be used to determine points and the line at infinity.
- A typical case is where three points $\mathbf{a}'$, $\mathbf{b}'$ and $\mathbf{c}'$ are identified on a line in an image.
- Suppose $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are the corresponding collinear points on the world line.
- The length ratio $d(\mathbf{a}, \mathbf{b}) : d(\mathbf{b}, \mathbf{c}) = a : b$ **is known**; $d(\mathbf{x}, \mathbf{y})$ is the Euclidean distance between points $\mathbf{x}$ and $\mathbf{y}$.

Computing a Vanishing Point from a Length Ratio



Solution:

i. Measure the distance ratio in the image, $d(\mathbf{a}',\mathbf{b}') : d(\mathbf{b}',\mathbf{c}') = a' : b'$.

ii. Points $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ may be represented as coordinates 0, a and $a+b$ in a coordinate frame on the line $\langle \mathbf{a}, \mathbf{b}, \mathbf{c} \rangle$.

These points are represented by homogeneous 2-vectors in $\mathbb{P}^1$, i.e. $(0,1)^\top$, $(a,1)^\top$ and $(a+b,1)^\top$.

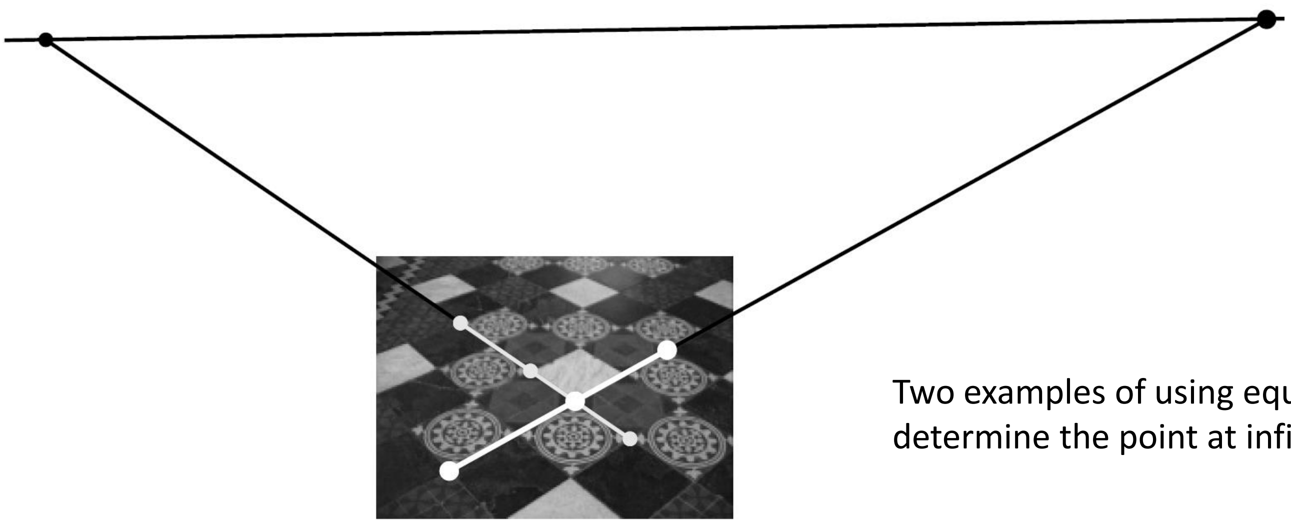
Similarly, $\langle \mathbf{a}', \mathbf{b}', \mathbf{c}' \rangle$ have coordinates $(0,1)^\top$, $(a',1)^\top$ and $(a'+b',1)^\top$.

Computing a Vanishing Point from a Length Ratio

Solution:

iii. Relative to these coordinate frames, compute the 1D projective transformation $\mathrm{H}_{2\times 2}$ mapping $\mathbf{a} \mapsto \mathbf{a}'$, $\mathbf{b} \mapsto \mathbf{b}'$ and $\mathbf{c} \mapsto \mathbf{c}'$.

iv. The image of the point at infinity (with coordinates $(1, 0)^\top$) under $\mathrm{H}_{2\times 2}$ is the vanishing point on the line $\langle \mathbf{a}', \mathbf{b}', \mathbf{c}' \rangle$.



Two examples of using equal length ratios on a line to determine the point at infinity.

Image source: “Multiple View Geometry in Computer Vision”, Richard Hartley and Andrew Zisserman

Circular Points and Their Dual

- Under any similarity transformation there are two points on $\mathbf{l}_\infty$ which are fixed.
- These are the circular points (also called the absolute points) $\mathbf{I}, \mathbf{J}$, with canonical coordinates:

$$\mathbf{I} = \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} \qquad \mathbf{J} = \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix}.$$

- The circular points are a pair of complex conjugate ideal points.

Circular Points and Their Dual

- The circular points, $\mathbf{I}, \mathbf{J}$, are fixed points under the projective transformation H if and only if H is a similarity, i.e.

$$\begin{aligned}\mathbf{I}' &= \mathrm{H_S}\mathbf{I} \\ &= \begin{bmatrix} s\cos\theta & -s\sin\theta & t_x \\ s\sin\theta & s\cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} \\ &= se^{-i\theta} \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} = \mathbf{I}\,,\end{aligned}$$

where $e^{i\theta} = \cos\theta + i\sin\theta$.

- With an analogous proof for $\mathbf{J}$.

- The converse is also true, i.e. if the circular points are fixed then the linear transformation is a similarity.

Circular Points and Their Dual

- The name "circular points" arises because every circle intersects $\mathbf{l}_\infty$ at the circular points.

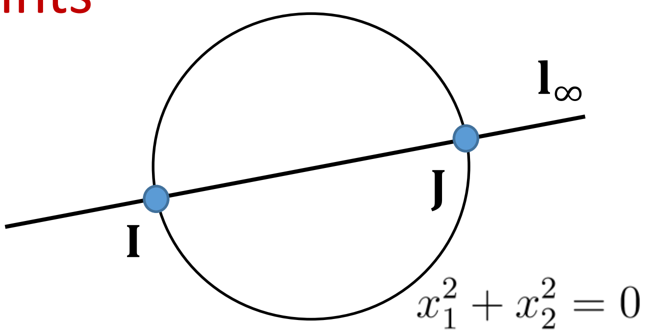
- To see this, we start from the conic equation of a circle, i.e. $a = c$ (we scale to 1) and $b = 0$:

$$x_1^2 + x_2^2 + dx_1x_3 + ex_2x_3 + fx_3^2 = 0$$

- This conic intersects $\mathbf{l}_\infty$ at the ideal points where $x_3 = 0$:

$$x_1^2 + x_2^2 = 0$$

$$\Rightarrow (x_1 + ix_2)(x_1 - ix_2) = 0$$

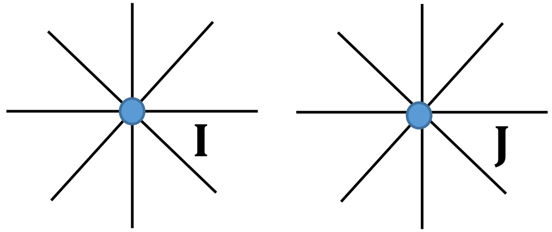


- with solution $\mathbf{I} = (1, \mathrm{i}, 0)^T, \mathbf{J} = (1, -\mathrm{i}, 0)^T$

Circular Points and Their Dual

- The dual to the circular points is the conic:

$$\mathrm{C}^*_\infty = \mathbf{I}\mathbf{J}^\mathsf{T} + \mathbf{J}\mathbf{I}^\mathsf{T}$$



- The conic C^*_∞ is a degenerate (rank 2) line conic which consists of the two circular points.

- In a Euclidean coordinate system, it is given by:

$$\mathrm{C}^*_\infty = \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} \begin{pmatrix} 1 & -i & 0 \end{pmatrix} + \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix} \begin{pmatrix} 1 & i & 0 \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Circular Points and Their Dual

- The conic C_∞^* is fixed under similarity transformations, i.e.

$$C_\infty^{*}{}' = H_S C_\infty^* H_S^\mathsf{T}$$

$$= \begin{pmatrix} s\cos\theta & -s\sin\theta & t_x \\ s\sin\theta & s\cos\theta & t_y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} s\cos\theta & s\sin\theta & 0 \\ -s\sin\theta & s\cos\theta & 0 \\ t_x & t_y & 1 \end{pmatrix}$$

$$= \begin{pmatrix} s\cos\theta & -s\sin\theta & 0 \\ s\sin\theta & s\cos\theta & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} s\cos\theta & s\sin\theta & 0 \\ -s\sin\theta & s\cos\theta & 0 \\ t_x & t_y & 1 \end{pmatrix}$$

$$= s \begin{pmatrix} \cos^2\theta + \sin^2\theta & 0 & 0 \\ 0 & \cos^2\theta + \sin^2\theta & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Circular Points and Their Dual

- Some properties of C^*_∞ in any projective frame:

i. C^*_∞ has 4 degrees of freedom:

A 3×3 homogeneous symmetric matrix has 5 degrees of freedom, but the constraint $\det(C^*_\infty) = 0$ reduces the degrees of freedom by 1.

ii. $\mathbf{l}_\infty$ is the null vector of C^*_∞:

This is clear from the definition: the circular points lie on $\mathbf{l}_\infty$, so that $\mathbf{I}^\top \mathbf{l}_\infty = \mathbf{J}^\top \mathbf{l}_\infty = 0$; then

$$C^*_\infty \mathbf{l}_\infty = (\mathbf{I}\mathbf{J}^\top + \mathbf{J}\mathbf{I}^\top)\mathbf{l}_\infty = \mathbf{I}(\mathbf{J}^\top \mathbf{l}_\infty) + \mathbf{J}(\mathbf{I}^\top \mathbf{l}_\infty) = \mathbf{0}.$$

Angles on the Projective Plane

- In Euclidean geometry, the angle between two lines is given by the inner product of the normals of $\mathbf{l} = (l_1, l_2, l_3)^\mathsf{T}$ and $\mathbf{m} = (m_1, m_2, m_3)^\mathsf{T}$:

$$\cos\theta = \frac{l_1 m_1 + l_2 m_2}{\sqrt{(l_1^2 + l_2^2)(m_1^2 + m_2^2)}}.$$

- **Problem with this expression**: it is not defined under projective transformation.
- Hence, the expression cannot be applied after an affine or projective transformation of the plane.

Angles on the Projective Plane

- Once the conic C^*_∞ is identified on the projective plane then Euclidean angles may be measured by

$$\cos\theta = \frac{\mathbf{l}^\mathsf{T} \mathtt{C}^*_\infty \mathbf{m}}{\sqrt{(\mathbf{l}^\mathsf{T} \mathtt{C}^*_\infty \mathbf{l})(\mathbf{m}^\mathsf{T} \mathtt{C}^*_\infty \mathbf{m})}},$$

- which is invariant to projective transformation.

Proof: We have $(\mathbf{l}' = \mathtt{H}^{-\mathsf{T}}\mathbf{l})$ and $(\mathtt{C}^{*\prime} = \mathtt{H}\mathtt{C}^*\mathtt{H}^\mathsf{T})$ under the point transformation $\mathbf{x}' = \mathrm{H}\mathbf{x}$, hence the numerator transforms as

$$\mathbf{l}^\mathsf{T} \mathtt{C}^*_\infty \mathbf{m} \mapsto \mathbf{l}^\mathsf{T} \mathtt{H}^{-1} \mathtt{H} \mathtt{C}^*_\infty \mathtt{H}^\mathsf{T} \mathtt{H}^{-\mathsf{T}} \mathbf{m} = \mathbf{l}^\mathsf{T} \mathtt{C}^*_\infty \mathbf{m}.$$

It can be verified that the denominator terms also stay the same, and the scales of $\mathbf{l}$ and $\mathbf{m}$ cancel out. $\square$

Angles on the Projective Plane

- Lines $\mathbf{l}$ and $\mathbf{m}$ are orthogonal if $\mathbf{l}^{\mathsf{T}} C_{\infty}^{*} \mathbf{m} = 0$.

Proof:

$$\cos\theta = \frac{\mathbf{l}^{\mathsf{T}} C_{\infty}^{*} \mathbf{m}}{\sqrt{(\mathbf{l}^{\mathsf{T}} C_{\infty}^{*} \mathbf{l})(\mathbf{m}^{\mathsf{T}} C_{\infty}^{*} \mathbf{m})}}$$

This is because $\cos\left(\frac{\pi}{2}\right) = 0$.

$\square$

Metric rectification using C_∞^*

- Once the conic C_∞^* is identified on the projective plane then projective distortion may be rectified up to a similarity.

Proof:

If the point transformation is $\mathbf{x}' = H\mathbf{x}$, we have

$$\begin{aligned}
{C_\infty^*}' &= (H_P\, H_A\, H_S)\, C_\infty^*\, (H_P\, H_A\, H_S)^\top = (H_P\, H_A)\underbrace{\left(H_S\, C_\infty^* H_S^\top\right)}_{= C_\infty^*}\left(H_A^\top\, H_P^\top\right) \\
&= (H_P\, H_A)\, C_\infty^*\left(H_A^\top\, H_P^\top\right) \\
&= \begin{bmatrix} KK^\top & KK^\top\mathbf{v} \\ \mathbf{v}^\top KK^\top & \mathbf{v}^\top KK^\top\mathbf{v} \end{bmatrix}.
\end{aligned}$$

It is clear that image of C_∞^* gives the projective ($\mathbf{v}$) and affine (K) components, but not the similarity component.

Recall: $$H = H_S\, H_A\, H_P = \begin{bmatrix} sR & \mathbf{t} \\ \mathbf{0}^\top & 1 \end{bmatrix}\begin{bmatrix} K & \mathbf{0} \\ \mathbf{0}^\top & 1 \end{bmatrix}\begin{bmatrix} I & \mathbf{0} \\ \mathbf{v}^\top & v \end{bmatrix} = \begin{bmatrix} A & \mathbf{t} \\ \mathbf{v}^\top & v \end{bmatrix}$$

Metric rectification using C_∞^*

- Given the identified C_∞^* in an image, i.e. C'^*_∞, a suitable rectifying homography H can be found from the SVD of C'^*_∞:

$$C_\infty^{*\prime} = U \begin{bmatrix} \sqrt{S} & 0 & 0 \\ 0 & \sqrt{S} & 0 \\ 0 & 0 & 0 \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{= C_\infty^*} \begin{bmatrix} \sqrt{S} & 0 & 0 \\ 0 & \sqrt{S} & 0 \\ 0 & 0 & 0 \end{bmatrix} U^\top$$

- where the rectifying projectivity is $H = U$ up to a similarity $\sqrt{S}$.
- S is the singular value of C'^*_∞.

Metric rectification using C_∞^*

- **Note:** In general C'^*_∞ does not fulfil the rank-2 and repeated singular value constraint due to noisy measurements, i.e.

$$C_\infty^{*\prime} = U \begin{bmatrix} S_1 & 0 & 0 \\ 0 & S_2 & 0 \\ 0 & 0 & S_3 \end{bmatrix} U^\top .$$

- We can simply set $S_3 = 0$, and the $S_2 = S_1$. This gives C, the closest rank-2 matrix with repeated singular values to the measured $C_\infty^{*\prime}$, i.e.

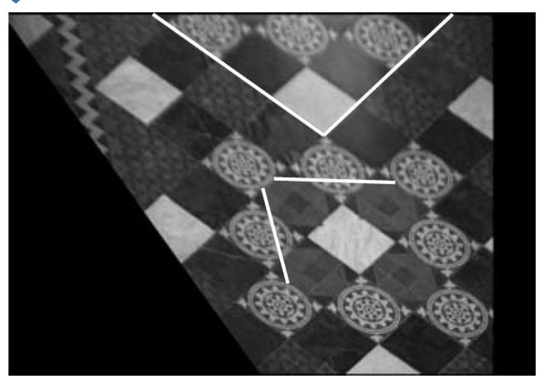
$$\underset{C}{\operatorname{argmin}} \left\| C_\infty^{*\prime} - C \right\|_F \text{ s.t rank}(C) = 2, \text{ and } S_1 = S_2 .$$

- $\|.\|_F$ denotes the Frobenius norm.

Identifying C^*_∞ in an Image

Example 1: Metric rectification of an affinely rectified image

1. Affine rectification, i.e. removal of projective distortion H_p (seen earlier)
2. Metric rectification, i.e. removal of affine distortion H_A



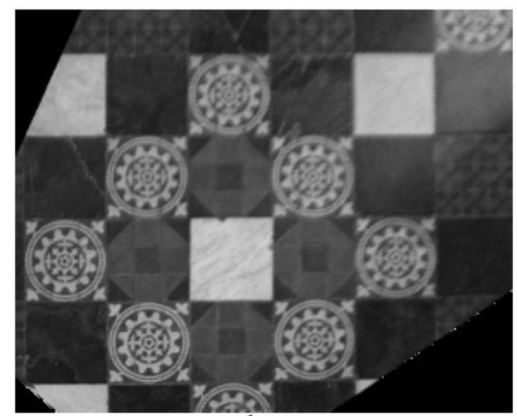


Image source: “Multiple View Geometry in Computer Vision”, Richard Hartley and Andrew Zisserman

Identifying C_∞^* in an Image

Example 1: Metric rectification of an affinely rectified image

- We have seen that

$$C_\infty^{*\,\prime} = (H_P H_A H_S) C_\infty^* (H_P H_A H_S)^\top = (H_P H_A) C_\infty^* \left(H_A^\top H_P^\top\right),$$

- which can be written as

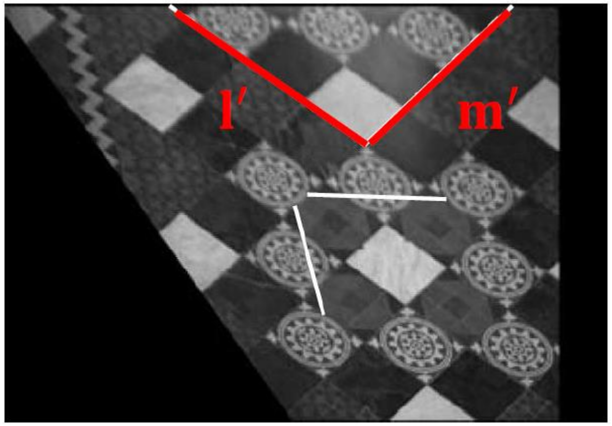
$$\underbrace{H_p^{-1} C'^*_\infty H_p^{-\top}}_{= C''^*_\infty} = H_A C_\infty^* H_A^\top,$$

- where C''^*_∞ is the image of the conic C_∞^* after removal of projective distortion.

Identifying C^*_∞ in an Image

Example 1: Metric rectification of an affinely rectified image

- We can compute C''^*_∞ and hence H_A from two pairs of orthogonal lines.
- Suppose the lines $\mathbf{l}'$, $\mathbf{m}'$ in the affinely rectified image correspond to an orthogonal line pair $\mathbf{l}$, $\mathbf{m}$ on the world plane, we get:



$$\underbrace{(\mathbf{l}^\top H_A^{-1})}_{\mathbf{l}'^\top}\underbrace{H_A C^*_\infty H_A^\top}_{C''^*_\infty}\underbrace{(H_A^{-\top}\mathbf{m})}_{\mathbf{m}'} = 0 \ , \qquad H_A = \begin{bmatrix} K & 0 \\ 0^\top & 1 \end{bmatrix}$$

$$\Rightarrow \begin{pmatrix} l'_1 & l'_2 & l'_3 \end{pmatrix} \begin{bmatrix} KK^\top & \mathbf{0} \\ \mathbf{0}^\top & 0 \end{bmatrix} \begin{pmatrix} m'_1 \\ m'_2 \\ m'_3 \end{pmatrix} = 0 \ ,$$

where we write $S_{2\times 2} = KK^\top$ with 3 independent elements.

Image source: “Multiple View Geometry in Computer Vision”, Richard Hartley and Andrew Zisserman

Identifying C_∞^* in an Image

Example 1: Metric rectification of an affinely rectified image

- Thus, the orthogonality constraint can be written as:

$$(l'_1 m'_1, l'_1 m'_2 + l'_2 m'_1, l'_2 m'_2)\,\mathbf{s} = 0,$$

where $\mathbf{s} = (s_{11}, s_{12}, s_{22})^\top$ is S written as a 3-vector.

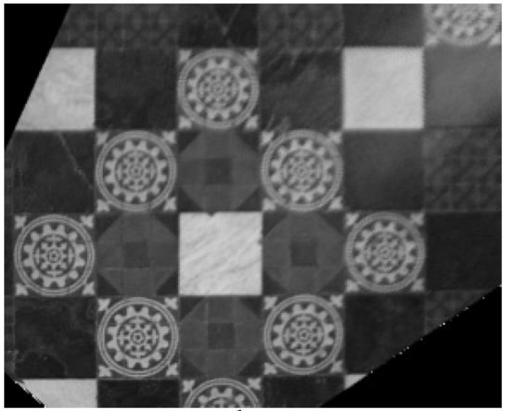
- Two constraints from two orthogonal line pairs which may be stacked to give a 2 × 3 matrix with **s** determined as the null vector.

- Thus S, and hence K (therefore H_A), is obtained up to scale by Cholesky decomposition.

Identifying C_∞^* in an Image

Example 2: Metric rectification of perspective image of the plane (not affinely rectified).

Removal of projective and affine distortion $H_p H_A$



This can be achieved by identifying C_∞^* on the perspective image, i.e.

$$C_\infty^{*}{}' = (H_P H_A) C_\infty^* (H_A^\top H_P^\top) = \begin{bmatrix} KK^\top & KK^\top \mathbf{v} \\ \mathbf{v}^\top KK^\top & \mathbf{v}^\top KK^\top \mathbf{v} \end{bmatrix}.$$

Image source: "Multiple View Geometry in Computer Vision", Richard Hartley and Andrew Zisserman

Identifying C^*_∞ in an Image

Example 2: Metric rectification of perspective image of the plane (not affinely rectified).

- Each orthogonal pair of lines $\mathbf{l}', \mathbf{m}'$ on the perspective image gives the constraint:

$$(l'_1 m'_1, (l'_1 m'_2 + l'_2 m'_1)/2, l'_2 m'_2, (l'_1 m'_3 + l'_3 m'_1)/2, (l'_2 m'_3 + l'_3 m'_2)/2, l'_3 m'_3)\, \mathbf{c} = 0$$

- where $\mathbf{c} = (a, b, c, d, e, f)^\mathsf{T}$ is C'^*_∞ written as a 6-vector.
- Five such constraints can be stacked to form a 5×6 matrix, and $\mathbf{c}$, and hence C'^*_∞ (therefore $\mathrm{H}_p\mathrm{H}_A$), is obtained as the null vector.

Stratification

- Note the two-step (remove projective then affine) and one-step (remove both) difference between example 1 and 2.

- The two-step approach is termed stratified.

The Plane at Infinity

- The plane at infinity has the canonical position $\boldsymbol{\pi}_\infty = (0, 0, 0, 1)^\top$ in affine 3-space.

- It contains the directions $\boldsymbol{D} = (X_1, X_2, X_3, 0)^\top$, and enables the identification of affine properties such as parallelism, particularly:

i. Two planes are parallel if, and only if, their line of intersection is on $\boldsymbol{\pi}_\infty$.

ii. A line is parallel to another line, or to a plane, if the point of intersection is on $\boldsymbol{\pi}_\infty$.

The Plane at Infinity

- The plane at infinity, $\boldsymbol{\pi}_\infty$, is a fixed plane under the projective transformation H if, and only if, H is an affinity, i.e.

$$\boldsymbol{\pi}'_\infty = \mathrm{H}_A^{-\top}\boldsymbol{\pi}_\infty = \begin{bmatrix} \mathrm{A}^{-\top} & \mathbf{0} \\ -\mathrm{t}^\top \mathrm{A}^{-\top} & 1 \end{bmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \boldsymbol{\pi}_\infty$$

- **Remarks:**

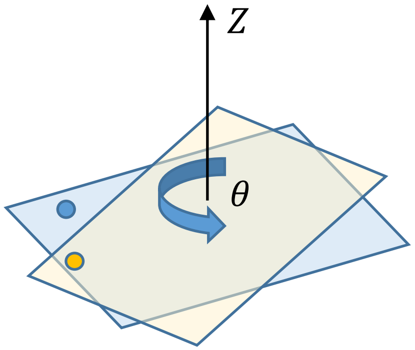
1. The plane $\boldsymbol{\pi}_\infty$ is, in general, only fixed as a set under an affinity; it is not fixed pointwise.

2. Under a particular affinity (for example a Euclidean motion) there may be planes in addition to $\boldsymbol{\pi}_\infty$ which are fixed. However, only $\boldsymbol{\pi}_\infty$ is fixed under any affinity.

The Plane at Infinity

Example: Consider the Euclidean transformation represented by the matrix

$$\mathrm{H_E} = \begin{bmatrix} \mathrm{R} & \mathbf{0} \\ \mathbf{0}^\mathsf{T} & 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$



- This is a rotation by θ about the Z-axis with a zero translation, hence, there is a pencil of fixed planes orthogonal to the z-axis.

- The planes are fixed as sets, but not pointwise as any (finite) point (not on the axis) is rotated in horizontal circles by this Euclidean action.

The Plane at Infinity

Example continue:

- Algebraically, the fixed planes of H are the eigenvectors of H^{T}, i.e.

$$\mathrm{H}^{-\mathsf{T}}\mathbf{v} = \lambda\mathbf{v} \;\Leftrightarrow\; \mathrm{H}^{-\mathsf{T}}\boldsymbol{\pi} = \lambda\boldsymbol{\pi},$$

- $\lambda, \mathbf{v}$ are the eigenvalues and eigenvectors of H^{T} and $\mathrm{H}^{-\mathsf{T}}$.

- In this case, the eigenvalues and eigenvectors of $\mathrm{H}_{\mathrm{E}}^{\mathsf{T}}$ are $\{e^{i\theta}, e^{-i\theta}, 1, 1\}$ and

$$\mathbf{E}_1 = \begin{pmatrix} 1 \\ i \\ 0 \\ 0 \end{pmatrix} \; \mathbf{E}_2 = \begin{pmatrix} 1 \\ -i \\ 0 \\ 0 \end{pmatrix} \; \mathbf{E}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \; \mathbf{E}_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

The Plane at Infinity

Example continue:

- The eigenvectors $\mathbf{E}_1$ and $\mathbf{E}_2$ are imaginary planes, and will not be discussed further.

- In addition to $\mathbf{E}_4$ (i.e. the plane at infinity), we can see that there is a pencil of fixed planes spanned by $\mathbf{E}_3$ and $\mathbf{E}_4$ under H_E, i.e.

$$\boldsymbol{\pi} = \mu \mathbf{E}_3 + \lambda \mathbf{E}_4.$$

- We say that the eigenvectors $\mathbf{E}_3$ and $\mathbf{E}_4$ are degenerate.

The Plane at Infinity

Example continue:

- The axis of this pencil is the line of intersection of the planes (perpendicular to the Z-axis) with $\boldsymbol{\pi}_\infty$, and the pencil includes $\boldsymbol{\pi}_\infty$, i.e.

$$\mathrm{L}^* = \begin{bmatrix} \mathbf{E}_3^\top \\ \mathbf{E}_4^\top \end{bmatrix}, \text{ with null-space basis } (1,0,0,0)^\top \text{ and } (0,1,0,0)^\top.$$

- $(1,0,0,0)^\top$ and $(0,1,0,0)^\top$ are ideal points that lie on $\boldsymbol{\pi}_\infty$, and hence $\mathbf{E}_3$ and $\mathbf{E}_4$ intersects at $\mathbf{l}_\infty$.

The Plane at Infinity

- We will see in Lecture 6 that uncalibrated two-view reconstructions lead to projective ambiguity.

- The identified π_∞ can be used to remove the projective ambiguity, where affine properties can be measured.

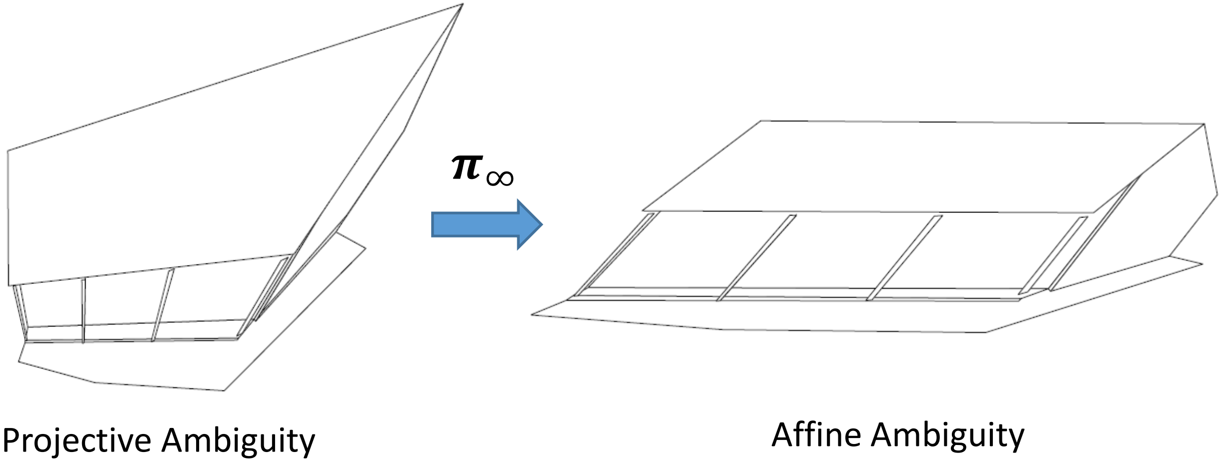


Image source: “Multiple View Geometry in Computer Vision”, Richard Hartley and Andrew Zisserman

The Absolute Conic

- The absolute conic, Ω_∞, is a (point) conic on $\boldsymbol{\pi}_\infty$.
- In a metric frame $\boldsymbol{\pi}_\infty = (0,0,0,1)^\mathsf{T}$, and points on Ω_∞ satisfy

$$\left.\begin{array}{r} X_1^2 + X_2^2 + X_3^2 \\ X_4 \end{array}\right\} = 0.$$

- Note that two equations are required to define Ω_∞.

The Absolute Conic

- For directions on $\boldsymbol{\pi}_\infty$ (i.e. points with $X_4 = 0$) the defining equation can be written

$$(\mathrm{X}_1, \mathrm{X}_2, \mathrm{X}_3)\mathtt{I}(\mathrm{X}_1, \mathrm{X}_2, \mathrm{X}_3)^\mathsf{T} = 0$$

- So that Ω_∞ corresponds to a conic C with matrix $\mathrm{C} = \mathtt{I}$; it is thus a conic of purely imaginary points on $\boldsymbol{\pi}_\infty$.

- The conic Ω_∞ is a geometric representation of the 5 additional degrees of freedom required to specify metric properties in an affine coordinate frame.

The Absolute Conic

- The absolute conic, Ω_∞ , is a fixed conic under the projective transformation H if, and only if, H is a similarity transformation.

Proof:

Since the absolute conic lies in $\boldsymbol{\pi}_\infty$, a transformation fixing it must fix $\boldsymbol{\pi}_\infty$, and hence must be affine, i.e.

$$\mathrm{H}_{\mathrm{A}} = \begin{bmatrix} \mathrm{A} & \mathbf{t} \\ \mathbf{0}^{\mathsf{T}} & 1 \end{bmatrix}.$$

At $\boldsymbol{\pi}_\infty$, $\Omega_\infty = \mathrm{I}_{3\times 3}$, and since it is fixed by H_A, one has $\mathrm{A}^{-\mathsf{T}}\mathrm{I}\mathrm{A}^{-1} = \mathrm{I}$ (up to scale), and taking inverses gives $\mathrm{A}\mathrm{A}^{\mathsf{T}} = \mathrm{I}$.

This means that A is orthogonal, hence a scaled rotation, or scaled rotation with reflection, i.e. similarity transform.

□

The Absolute Conic

- Even though Ω_∞ does not have any real points, it shares the properties of any conic:

1. The conic Ω_∞ is only fixed as a set by a general similarity; it is not fixed pointwise.

   Remark: This means that under a similarity a point on Ω_∞ may travel to another point on Ω_∞, but it is not mapped to a point off the conic.

The Absolute Conic

2. All circles intersect Ω_∞ in two points.

   Remark: Suppose the support plane of the circle is $\boldsymbol{\pi}$. Then $\boldsymbol{\pi}$ intersects $\boldsymbol{\pi}_\infty$ in a line, and this line intersects Ω_∞ in two points. These two points are the circular points of $\boldsymbol{\pi}$.

3. All spheres intersect $\boldsymbol{\pi}_\infty$ in Ω_∞ .

The Absolute Conic

- The angle between two lines with directions (3-vectors) $\mathbf{d}_1$ and $\mathbf{d}_2$ is given by:

$$\cos\theta = \frac{(\mathbf{d}_1^\top \Omega_\infty \mathbf{d}_2)}{\sqrt{(\mathbf{d}_1^\top \Omega_\infty \mathbf{d}_1)(\mathbf{d}_2^\top \Omega_\infty \mathbf{d}_2)}}$$

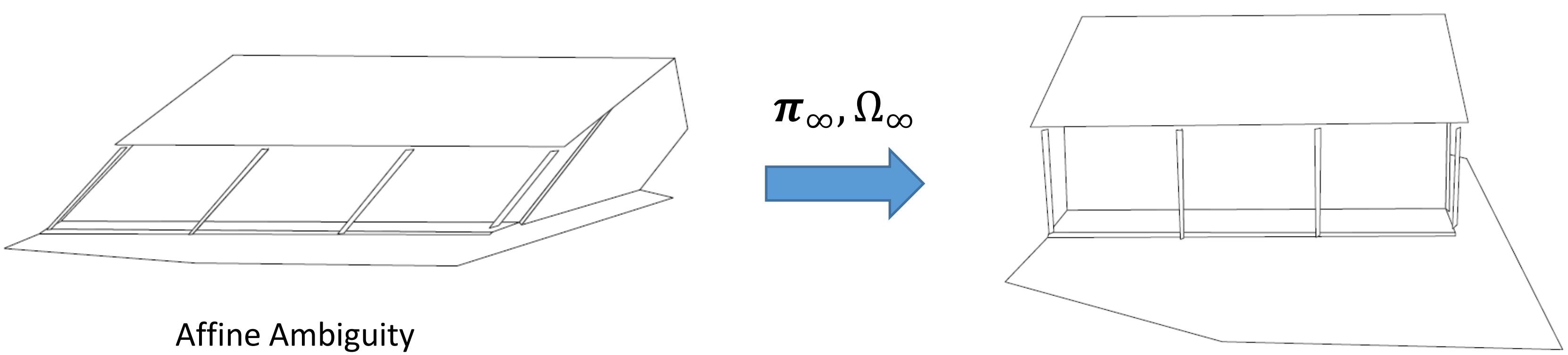
- where $\mathbf{d}_1$ and $\mathbf{d}_2$ are the points of intersection of the lines with the plane $\boldsymbol{\pi}_\infty$ containing the conic Ω_∞.

- And Ω_∞ is the matrix representation of the absolute conic in that plane.

- Two directions $\mathbf{d}_1$ and $\mathbf{d}_2$ are orthogonal if $\mathbf{d}_1^\top \Omega_\infty \mathbf{d}_2 = 0$.

The Absolute Conic: Orthogonality and Polarity

- We will see in Lecture 5 that the imaged absolute conic can be used to recover the camera intrinsics, i.e. calibration.
- Furthermore, we will see in Lecture 6 that both the absolute conic and plane at infinity can be used to remove affine distortion, hence the metric properties can be measured.



Affine Ambiguity

Similarity Ambiguity

Image source: "Multiple View Geometry in Computer Vision", Richard Hartley and Andrew Zisserman

The Absolute Dual Quadric

- The dual of the absolute conic Ω_∞ is a degenerate dual quadric in 3-space called the absolute dual quadric, and denoted by Q^*_∞.

- **Geometrically** Q^*_∞ consists of the planes tangent to Ω_∞, so that Ω_∞ is the "rim" of Q^*_∞, hence called a rim quadric.

- **Algebraically** Q^*_∞ is represented by a 4 × 4 homogeneous matrix of rank 3, with the canonical form:

$$Q^*_\infty = \begin{bmatrix} I & \mathbf{0} \\ \mathbf{0}^\top & 0 \end{bmatrix}.$$

The Absolute Dual Quadric

- The dual quadric Q_∞^* is a degenerate quadric.

- There are 8 degrees of freedom (a symmetric matrix has 10 independent elements, but the irrelevant scale and zero determinant).

The Absolute Dual Quadric

- The absolute dual quadric, Q_∞^*, is fixed under the projective transformation H if, and only if, H is a similarity.

Proof:

Since Q_∞^* is a dual quadric, it is fixed under H if and only if $\mathrm{Q}_\infty^* = \mathrm{H}\mathrm{Q}_\infty^*\mathrm{H}^\mathsf{T}$. Applying this with an arbitrary transform

$\mathrm{H} = \begin{bmatrix} \mathtt{A} & \mathbf{t} \\ \mathbf{v}^\mathsf{T} & k \end{bmatrix}$, we get $\begin{bmatrix} \mathtt{I} & \mathbf{0} \\ \mathbf{0}^\mathsf{T} & 0 \end{bmatrix} = \begin{bmatrix} \mathtt{A} & \mathbf{t} \\ \mathbf{v}^\mathsf{T} & k \end{bmatrix} \begin{bmatrix} \mathtt{I} & \mathbf{0} \\ \mathbf{0}^\mathsf{T} & 0 \end{bmatrix} \begin{bmatrix} \mathtt{A}^\mathsf{T} & \mathbf{v} \\ \mathbf{t}^\mathsf{T} & k \end{bmatrix}$

$$= \begin{bmatrix} \mathtt{A}\mathtt{A}^\mathsf{T} & \mathtt{A}\mathbf{v} \\ \mathbf{v}^\mathsf{T}\mathtt{A}^\mathsf{T} & \mathbf{v}^\mathsf{T}\mathbf{v} \end{bmatrix}$$

The Absolute Dual Quadric

Proof (continued):

$$\begin{bmatrix} \mathrm{I} & \mathbf{0} \\ \mathbf{0}^\mathsf{T} & 0 \end{bmatrix} = \begin{bmatrix} \mathrm{A}\mathrm{A}^\mathsf{T} & \mathrm{A}\mathbf{v} \\ \mathbf{v}^\mathsf{T}\mathrm{A}^\mathsf{T} & \mathbf{v}^\mathsf{T}\mathbf{v} \end{bmatrix}$$

which must be true up to scale.

By inspection, this equation holds if and only if $\mathbf{v} = \mathbf{0}$ and A is a scaled orthogonal matrix (scaling, rotation and possible reflection).

In other words, H is a similarity transform.

□

The Absolute Dual Quadric

- The plane at infinity $\boldsymbol{\pi}_\infty$ is the null-vector of Q_∞^*.

Remarks:

This is easily verified when Q_∞^* has its canonical form in a metric frame since then, with $\boldsymbol{\pi}_\infty = (0,0,0,1)^\top$, $\mathrm{Q}_\infty^* \boldsymbol{\pi}_\infty = \mathbf{0}$.

This property holds in any frame as may be readily seen algebraically from the transformation properties of planes and dual quadrics: if $\mathbf{X}' = \mathrm{H}\mathbf{X}$, then ${\mathrm{Q}_\infty^*}' = \mathrm{H}\mathrm{Q}_\infty^* \mathrm{H}^\top$, ${\boldsymbol{\pi}_\infty}' = \mathrm{H}^{-\top}\boldsymbol{\pi}_\infty$, and

$${\mathrm{Q}_\infty^*}'\boldsymbol{\pi}'_\infty = (\mathrm{H}\,\mathrm{Q}_\infty^*\,\mathrm{H}^\top)\mathrm{H}^{-\top}\boldsymbol{\pi}_\infty = \mathrm{H}\mathrm{Q}_\infty^*\boldsymbol{\pi}_\infty = \mathbf{0}.$$

The Absolute Dual Quadric

- The angle between two planes $\boldsymbol{\pi}_1$ and $\boldsymbol{\pi}_2$ is given by

$$\cos\theta = \frac{\boldsymbol{\pi}_1^{\mathsf{T}} \mathrm{Q}_\infty^* \boldsymbol{\pi}_2}{\sqrt{\left(\boldsymbol{\pi}_1^{\mathsf{T}} \mathrm{Q}_\infty^* \boldsymbol{\pi}_1\right)\left(\boldsymbol{\pi}_2^{\mathsf{T}} \mathrm{Q}_\infty^* \boldsymbol{\pi}_2\right)}}.$$

Proof:

Consider two planes with Euclidean coordinates $\boldsymbol{\pi}_1 = (\boldsymbol{n}_1^{\mathsf{T}}, d_1)^{\mathsf{T}}$, $\boldsymbol{\pi}_2 = (\boldsymbol{n}_2^{\mathsf{T}}, d_2)^{\mathsf{T}}$. In a Euclidean frame, Q_∞^* has the form

$\mathrm{Q}_\infty^* = \begin{bmatrix} \mathrm{I} & \mathbf{0} \\ \mathbf{0}^{\mathsf{T}} & 0 \end{bmatrix}$, and we get $\cos\theta = \dfrac{\mathbf{n}_1^{\mathsf{T}} \mathbf{n}_2}{\sqrt{\left(\mathbf{n}_1^{\mathsf{T}} \mathbf{n}_1\right)\left(\mathbf{n}_2^{\mathsf{T}} \mathbf{n}_2\right)}}$

which is the angle between the planes expressed in terms of a scalar product of their normals.

The Absolute Dual Quadric

Remarks:

If the planes and Q^*_∞ are projectively transformed,

$$\cos\theta = \frac{\boldsymbol{\pi}_1^\mathsf{T} \mathsf{Q}^*_\infty \boldsymbol{\pi}_2}{\sqrt{\left(\boldsymbol{\pi}_1^\mathsf{T} \mathsf{Q}^*_\infty \boldsymbol{\pi}_1\right)\left(\boldsymbol{\pi}_2^\mathsf{T} \mathsf{Q}^*_\infty \boldsymbol{\pi}_2\right)}}.$$

will still determine the angle between planes due to the (covariant) transformation properties of planes and dual quadrics.

Exercise: Prove it!

Summary

- Students should be able to:

1. Represent points, planes, lines and quadrics in $\mathbb{P}^3$.
2. Use line at infinity and/or circular points to remove affine and/or projective distortions.
3. Describe the plane at infinity and its invariance under affine transformation.
4. Describe the absolute conic (and its absolute dual quadrics) and its invariance under similarity transformation.