NUS National University of Singapore | School of Computing

CS4277 / CS5477
3D Computer Vision

Lecture 6:
The fundamental and essential matrices

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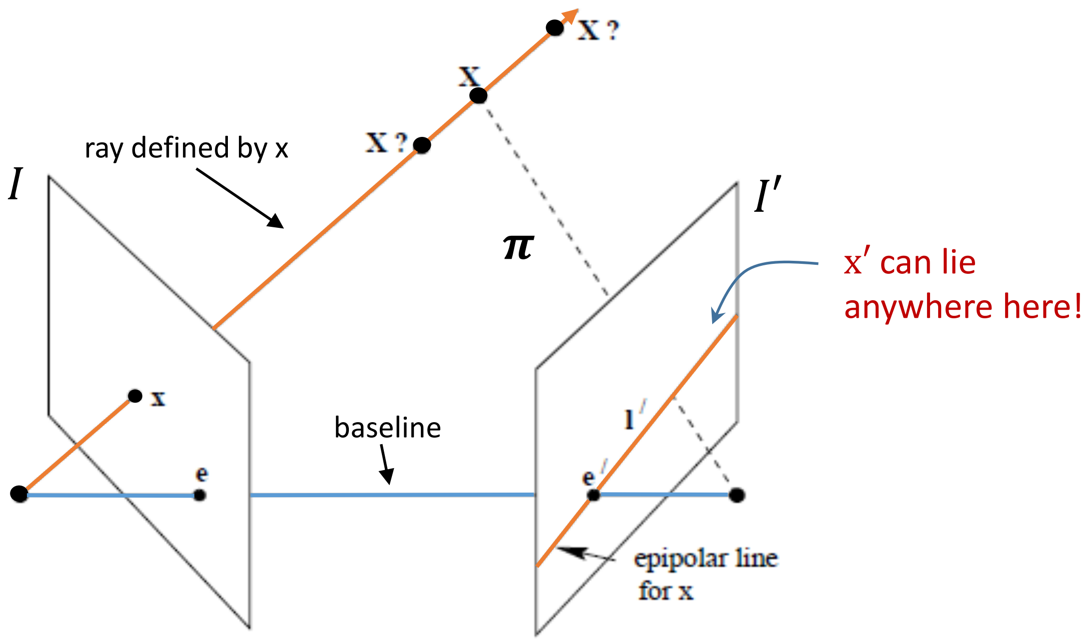
Semester 2

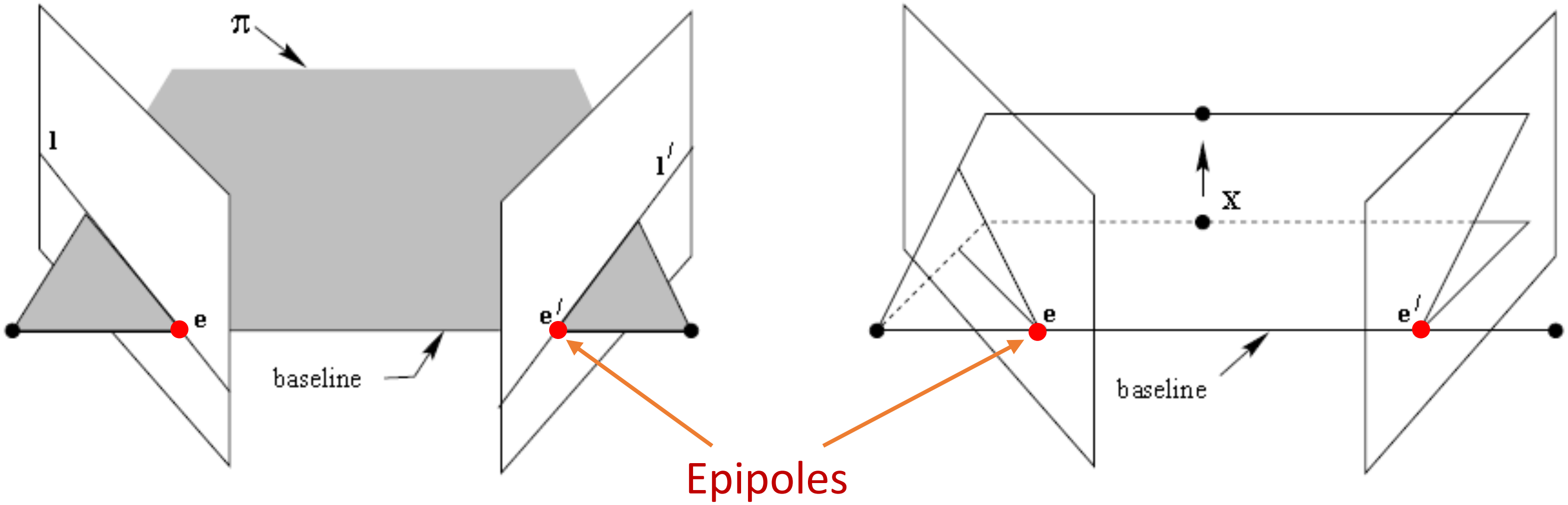
The Epipolar Geometry



- The image point x in I back-projects to a ray, and this ray projects to I' as the epipolar line $\mathbf{l}'$.
- The corresponding point $\mathbf{x}'$ can lie anywhere on $\mathbf{l}'$.
- Epipolar plane $\boldsymbol{\pi}$ is determined by the baseline and ray defined by $\mathbf{x}$.

Image Source: R. Hartley and A. Zisserman, "Multiple View Geometry in Computer Vision"

The Epipolar Geometry: Terminology

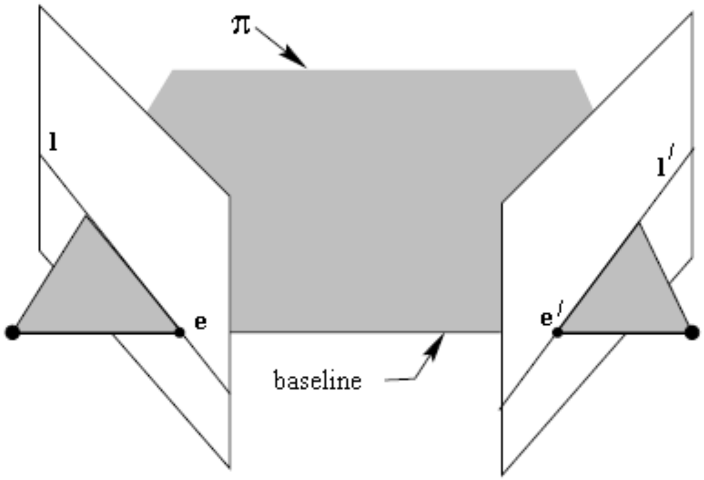


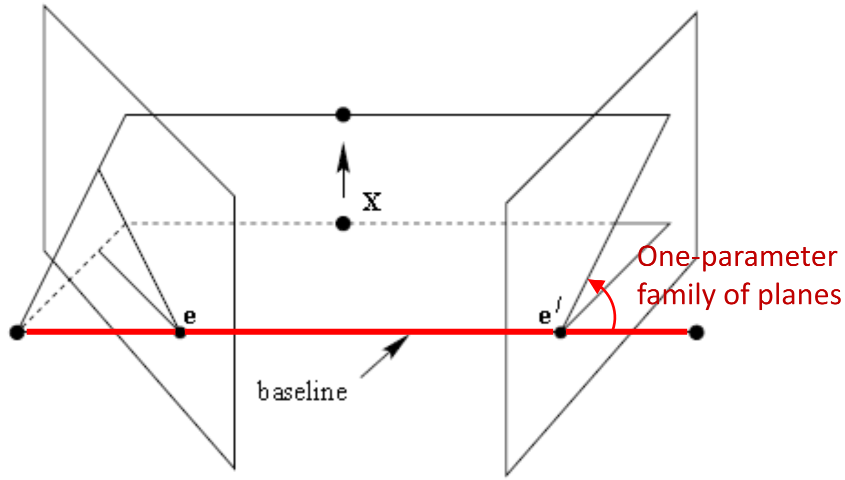
Epipoles (e, e′):

- Point of intersection of the line joining the camera centers (baseline) with the image plane.
- Equivalently, it is the image in one view of the camera center of the other view.
- Also, the vanishing point of the baseline (translation) direction.

Image Source: R. Hartley and A. Zisserman, "Multiple View Geometry in Computer Vision"

The Epipolar Geometry: Terminology



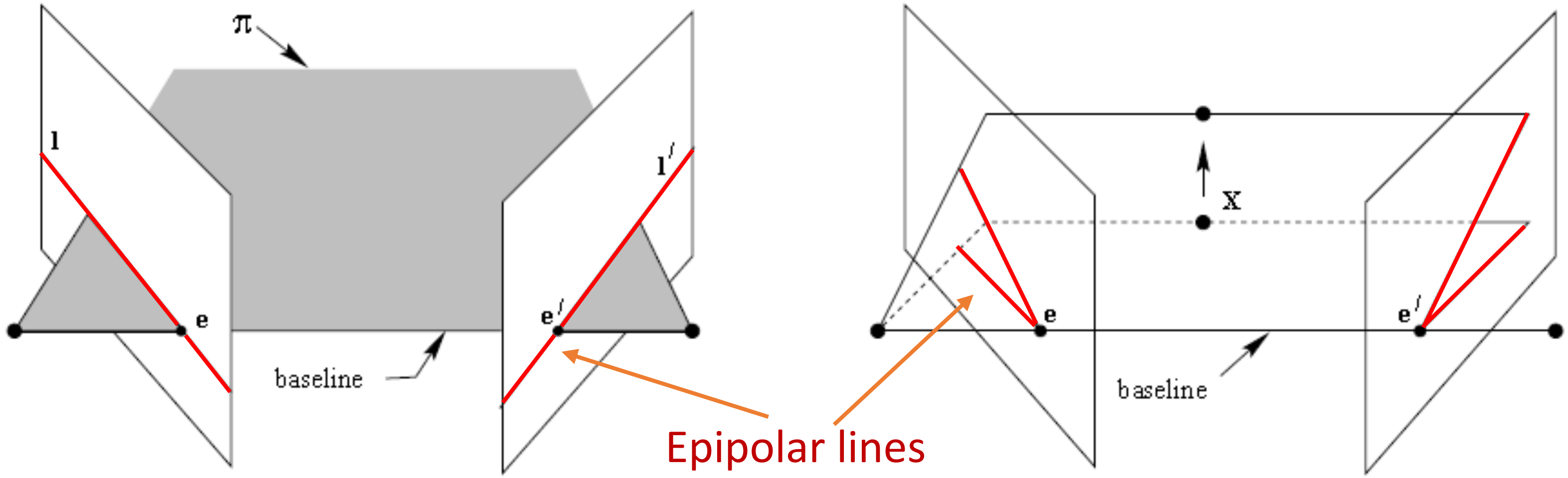


Epipolar plane π :

- A plane containing the baseline.
- There is a one-parameter family (a pencil) of epipolar planes.

Image Source: R. Hartley and A. Zisserman, "Multiple View Geometry in Computer Vision"

The Epipolar Geometry: Terminology



Epipolar lines $(\mathbf{l}, \mathbf{l}')$:

- The intersection of an epipolar plane with the image plane.
- All epipolar lines intersect at the epipole.
- An epipolar plane intersects the left and right image plane in epipolar lines, and defines the correspondences between the lines.

Image Source: R. Hartley and A. Zisserman, "Multiple View Geometry in Computer Vision"

The Fundamental Matrix

- The fundamental matrix is the algebraic representation of epipolar geometry.

- Gives the projective mapping relationship between a point $\mathbf{x}$ on one image to a line $\mathbf{l}'$ on the other.

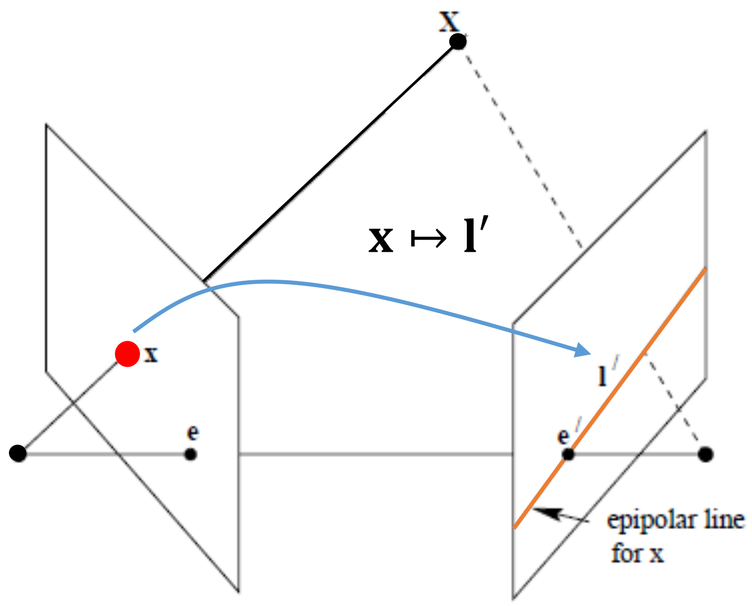


Image Source: R. Hartley and A. Zisserman, "Multiple View Geometry in Computer Vision"

F Matrix: Geometric Derivation

- The mapping $\mathbf{x} \mapsto \mathbf{l}'$ may be decomposed into two steps:

1. The point $\mathbf{x}$ is mapped to some point $\mathbf{x}'$ in the other image lying on the epipolar line $\mathbf{l}'$; this point $\mathbf{x}'$ is a potential match for the point $\mathbf{x}$.

2. The epipolar line $\mathbf{l}'$ is obtained as the line joining $\mathbf{x}'$ to the epipole $\mathbf{e}'$.

F Matrix: Geometric Derivation

Step 1: Point transfer via a plane.

- Consider a plane $\boldsymbol{\pi}$ in space not passing through either of the two camera centres and contains the point $\mathbf{X}$.
- Thus, there is a 2D homography $\mathrm{H}_{\boldsymbol{\pi}}$ mapping each $\mathbf{x}_i$ to $\mathbf{x}'_i$.

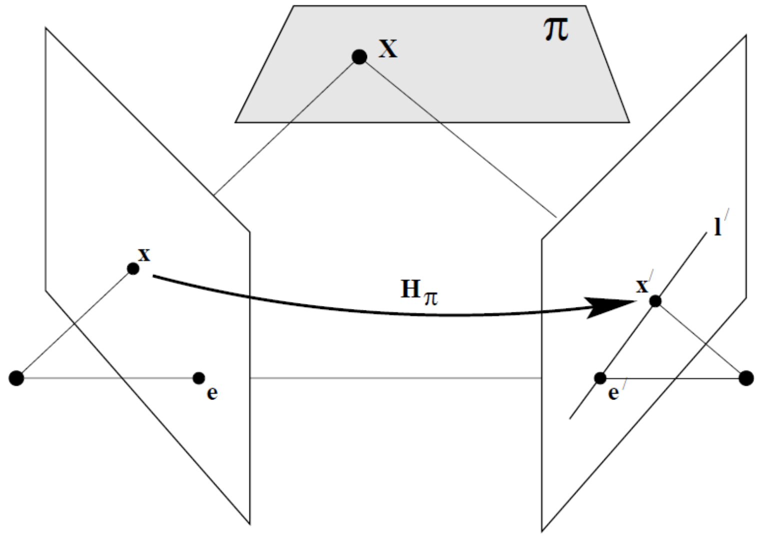


Image Source: R. Hartley and A. Zisserman, "Multiple View Geometry in Computer Vision"

F Matrix: Geometric Derivation

Step 2: Constructing the epipolar line.

- Given the point $\mathbf{x}'$, the epipolar line $\mathbf{l}'$ passing through $\mathbf{x}'$ and the epipole $\mathbf{e}'$ can be written as $\mathbf{l}' = \mathbf{e}' \times \mathbf{x}' = [\mathbf{e}']_\times \mathbf{x}'$.
- Since $\mathbf{x}'$ may be written as $\mathbf{x}' = \mathrm{H}_{\boldsymbol{\pi}}\mathbf{x}$, we have:

$$\mathbf{l}' = [\mathbf{e}']_\times \mathrm{H}_{\boldsymbol{\pi}} \mathbf{x} = \mathrm{F}\mathbf{x} \ ,$$

where we define $\mathrm{F} = [\mathbf{e}']_\times \mathrm{H}_{\boldsymbol{\pi}}$ as the fundamental matrix.

Cross-Product as Matrix Multiplication

- Vector cross product can be expressed as the product of a skew-symmetric matrix and a vector:

$$\mathbf{a} \times \mathbf{b} = [\mathbf{a}]_\times \mathbf{b} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

F Matrix: Geometric Derivation

- The fundamental matrix F may be written as:

$$\mathrm{F} = [\mathbf{e}']_{\times} \mathrm{H}_{\boldsymbol{\pi}},$$

- where $\mathrm{H}_{\boldsymbol{\pi}}$ is the transfer mapping from one image to another via any plane.

- Furthermore, since $[\mathbf{e}']_{\times}$ has rank 2 and $\mathrm{H}_{\boldsymbol{\pi}}$ rank 3, F is a matrix of rank 2.

F Matrix: Geometric Derivation

- Geometrically, F represents a mapping from the 2-dimensional projective plane $\mathbb{P}^2$ of the first image to the pencil of epipolar lines through the epipole $\mathbf{e}'$.

- Thus, it represents a mapping of $\mathbb{P}^2 \mapsto \mathbb{P}^1$, and hence must have rank 2.

- **Note:** The plane is simply used here as a means of defining a point map from one image to another, but not required for F to exist.

F Matrix: Algebraic Derivation

- The form of the fundamental matrix in terms of the two camera projection matrices, P and P', may be derived algebraically.

- The back-projected ray from $\mathbf{x}$ is given by:

$$\mathbf{X}(\lambda) = \mathrm{P}^{+}\mathbf{x} + \lambda\mathbf{C} \, ,$$

where

- P^{+} is the pseudo-inverse of P, i.e. $\mathrm{PP}^{+} = \mathrm{I}$
- $\mathbf{C}$ the null-vector of P, i.e. the camera center, $\mathrm{P}\mathbf{C} = \mathbf{0}$
- The ray is parametrized by the scalar λ

F Matrix: Algebraic Derivation

- Let's consider two points on the ray: $\mathrm{P}^{+}\mathbf{x}$ at $\lambda = 0$ and the first camera center $\mathbf{C}$ at $\lambda = \infty$.

- These two points are imaged by the second camera P' at $\mathrm{P}'\mathrm{P}^{+}\mathbf{x}$ and $\mathrm{P}'\mathbf{C}$, respectively in the second view.

- The epipolar line is the line joining these two projected points, i.e. $\mathbf{l}' = (\mathrm{P}'\mathbf{C}) \times (\mathrm{P}'\mathrm{P}^{+}\mathbf{x})$.

- The point $\mathrm{P}'\mathbf{C}$ is the epipole in the second image, i.e. $\mathbf{e}'$.

F Matrix: Algebraic Derivation

- Thus, $\mathbf{l}' = [\mathbf{e}']_\times (\mathtt{P}'\mathtt{P}^+)\mathbf{x} = \mathtt{F}\mathbf{x}$, where F is the matrix

$$\mathtt{F} = [\mathbf{e}']_\times \mathtt{P}'\mathtt{P}^+.$$

- This is similar to $\mathtt{F} = [\mathbf{e}']_\times \mathtt{H}_{\boldsymbol{\pi}}$ that we have derived geometrically.

- We can see that the homography takes the form

$$\mathtt{H}_{\boldsymbol{\pi}} = \mathtt{P}'\mathtt{P}^+$$

in terms of the two camera matrices.

F Matrix: Algebraic Derivation

- **Remarks:** Note that this derivation breaks down in the case where the two camera centres are the same.

Proof:

$\mathbf{e}' = \mathrm{P}'\mathbf{C} = \mathbf{0}$**,** when $\mathbf{C} = \mathbf{C}'$. It follows that:

$$\mathrm{F} = [\mathrm{e}']_\times \mathrm{P}'\mathrm{P}^+ = \mathbf{0}. \qquad \square$$

F Matrix: Algebraic Derivation

Example: Suppose the camera matrices are those of a calibrated stereo rig with the world origin at the first camera

$$\mathrm{P} = \mathrm{K}[\mathrm{I} \mid \mathbf{0}] \qquad \mathrm{P}' = \mathrm{K}'[\mathrm{R} \mid \mathbf{t}].$$

Then

$$\mathrm{P}^{+} = \begin{bmatrix} \mathrm{K}^{-1} \\ \mathbf{0}^{\mathsf{T}} \end{bmatrix} \qquad \mathbf{C} = \begin{pmatrix} \mathbf{0} \\ 1 \end{pmatrix}$$

and

$$\begin{aligned} \mathrm{F} &= [\mathrm{P}'\mathbf{C}]_{\times}\mathrm{P}'\mathrm{P}^{+} \\ &= [\mathrm{K}'\mathbf{t}]_{\times}\mathrm{K}'\mathrm{R}\mathrm{K}^{-1} = \mathrm{K}'^{-\mathsf{T}}[\mathbf{t}]_{\times}\mathrm{R}\mathrm{K}^{-1} = \mathrm{K}'^{-\mathsf{T}}\mathrm{R}[\mathrm{R}^{\mathsf{T}}\mathbf{t}]_{\times}\mathrm{K}^{-1} = \mathrm{K}'^{-\mathsf{T}}\mathrm{R}\mathrm{K}^{\mathsf{T}}[\mathrm{K}\mathrm{R}^{\mathsf{T}}\mathbf{t}]_{\times} \end{aligned}$$

Correspondence Condition

- For any pair of corresponding points $\mathbf{x} \leftrightarrow \mathbf{x}'$ in two images:

$$\mathbf{x}'^{\top} \mathrm{F} \mathbf{x} = 0$$

Proof:

$\mathbf{x}'$ lies on the epipolar line $\mathbf{l}' = \mathrm{F}\mathbf{x}$ corresponding to the point $\mathbf{x}$

$$\Rightarrow 0 = \mathbf{x}'^{\top} \mathbf{l}' = \mathbf{x}'^{\top} \mathrm{F} \mathbf{x} = 0$$

Correspondence Condition

- The importance of the relation $\mathbf{x}'^{\top}\mathrm{F}\mathbf{x} = 0$ is that it gives a way of characterizing the fundamental matrix without reference to the camera matrices.
- That is the relation is only in terms of corresponding image points, and this enables F to be computed from image correspondences alone.
- We will discuss the details later: how many correspondences are required to compute F from $\mathbf{x}'^{\top}\mathrm{F}\mathbf{x} = 0$?

Properties of the F Matrix

- **Transpose**:
  - F is the fundamental matrix of the pair of cameras $(\mathrm{P}, \mathrm{P}')$
  - $\mathrm{F}^\top$ is the fundamental matrix of the pair in the opposite order: $(\mathrm{P}', \mathrm{P})$
- **Epipolar lines**:
  - For any point $\mathbf{x}$ in first image, corresponding epipolar line is $\mathbf{l}' = \mathrm{F}\mathbf{x}$
  - $\mathbf{l} = \mathrm{F}^\top\mathbf{x}'$ represents epipolar line corresponding to $\mathbf{x}'$ in second image
- **Epipole**:
  - For any point $\mathbf{x}$ (other than $\mathbf{e}$) the epipolar line $\mathbf{l}' = \mathrm{F}\mathbf{x}$ contains the epipole $\mathbf{e}'$
  - $\mathbf{e}'$ satisfies $\mathbf{e}'^\top(\mathrm{F}\mathbf{x}) = (\mathbf{e}'^\top\mathrm{F})\mathbf{x} = 0$ for all $\mathbf{x}$
  - $\mathbf{e}'^\top\mathrm{F} = \mathbf{0}$, i.e. $\mathbf{e}'$ is the left null-vector of F
  - $\mathrm{F}\mathbf{e} = \mathbf{0}$, i.e. $\mathbf{e}$ is the right null-vector of F

Properties of the F Matrix

- **7 degrees of freedom (9 elements – 2 dof)**:
  - 3 x 3 homogenoueous matrix with 8 independent ratios ⇒ -1 dof
  - $\det(\mathrm{F}) = 0$ ⇒ -1 dof
- **Not a proper correlation (not invertible)**:
  - Projective map taking a point to a line
  - A point in first image $\mathbf{x}$ defines a line in the second $\mathbf{l} = \mathrm{F}\mathbf{x}$, i.e. epipolar line of $\mathbf{x}$
  - If $\mathbf{l}$ and $\mathbf{l}'$ are corresponding epipolar lines, then any point $\mathbf{x}$ on $\mathbf{l}$ is mapped to the same line $\mathbf{l}'$
  - This means no inverse mapping, and F is not of full rank

Summary of F Matrix Properties

- F is a rank 2 homogeneous matrix with 7 degrees of freedom.
- **Point correspondence**: If x and x' are corresponding image points, then $x'^{\top}Fx = 0$.
- **Epipolar lines**:
  - ◇ $l' = Fx$ is the epipolar line corresponding to x.
  - ◇ $l = F^{\top}x'$ is the epipolar line corresponding to x'.
- **Epipoles**:
  - ◇ $Fe = 0$.
  - ◇ $F^{\top}e' = 0$.
- **Computation from camera matrices** P, P':
  - ◇ General cameras,
    $F = [e']_{\times}P'P^{+}$, where P^{+} is the pseudo-inverse of P, and $e' = P'C$, with $PC = 0$.
  - ◇ Canonical cameras, $P = [I \mid 0]$, $P' = [M \mid m]$,
    $F = [e']_{\times}M = M^{-\top}[e]_{\times}$, where $e' = m$ and $e = M^{-1}m$.
  - ◇ Cameras not at infinity $P = K[I \mid 0]$, $P' = K'[R \mid t]$,
    $F = K'^{-\top}[t]_{\times}RK^{-1} = [K't]_{\times}K'RK^{-1} = K'^{-\top}RK^{\top}[KR^{\top}t]_{\times}$.

Source: Page 246, R. Hartley and A. Zisserman, "Multiple View Geometry in Computer Vision"

The Epipolar Line Homography

- Suppose $\mathbf{l}$ and $\mathbf{l}'$ are corresponding epipolar lines, and $\mathbf{k}$ is any line not passing through the epipole $\mathbf{e}$, then $\mathbf{l}$ and $\mathbf{l}'$ are related by $\mathbf{l}' = \mathrm{F}[\mathbf{k}]_\times \mathbf{l}$, where $\mathrm{F}[\mathbf{k}]_\times$ is a homography.

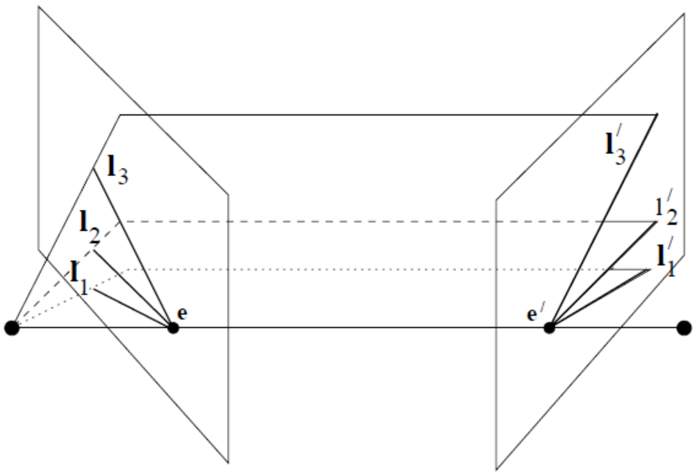
- Symmetrically, $\mathbf{l} = \mathrm{F}^\top[\mathbf{k}'] \times \mathbf{l}'$.

Proof:

The expression $[\mathbf{k}]_\times \mathbf{l} = \mathbf{k} \times \mathbf{l}$ is the point of intersection of the two lines **k** and **l**, and hence a point on the epipolar line $\mathbf{l}$ – call it $\mathbf{x}$.

Hence, $\mathrm{F}[\mathbf{k}]_\times \mathbf{l} \;=\; \mathrm{F}\mathbf{x}$ is the epipolar line corresponding to the point $\mathbf{x}$, namely the line $\mathbf{l}$.

The Epipolar Line Homography



Homography in $\mathbb{P}^2$:

$$\mathbf{l}' = \mathrm{F}[\mathbf{k}]_\times \mathbf{l}$$

$$\mathbf{l} = \mathrm{F}^\top [\mathbf{k}'] \times \mathbf{l}'$$

- There is a pencil of epipolar lines in each image centred on the epipole.
- The correspondence between epipolar lines, $\mathbf{l}_i \leftrightarrow \mathbf{l}_i$, is defined by the pencil of planes with axis the baseline.

Source: Page 246, R. Hartley and A. Zisserman, “Multiple View Geometry in Computer Vision”

Special Motion: Pure Translation

- Suppose the motion of the cameras is a pure translation with no rotation ($\mathrm{R} = \mathrm{I}$) and no change in the internal parameters ($\mathrm{K} = \mathrm{K}'$).

- The two cameras are $\mathrm{P} = \mathrm{K}[\mathrm{I} \mid \mathbf{0}]$ and $\mathrm{P}' = \mathrm{K}[\,\mathrm{I} \mid \mathbf{t}\,]$, and

$$\mathrm{F} = [\mathbf{e}']_\times \mathrm{K}\mathrm{K}^{-1} = [\mathbf{e}']_\times \,.$$

- F is skew-symmetric and has only 2 degrees of freedom, which correspond to the position of the epipole.

Special Motion: Pure Translation

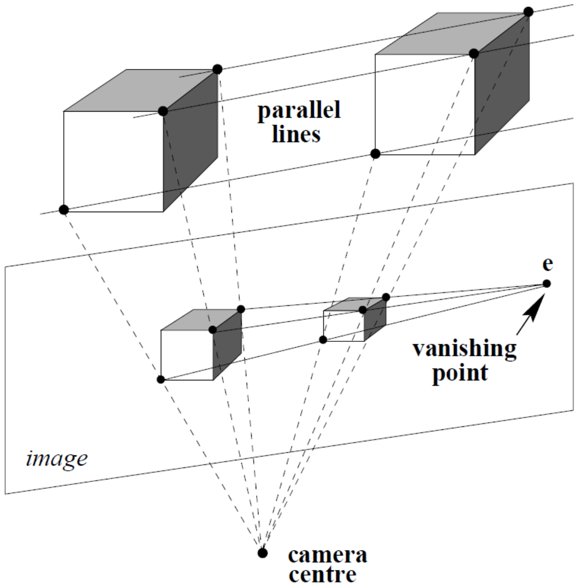
- The epipolar line of $\mathbf{x}$ is $\mathbf{l}' = \mathrm{F}\mathbf{x} = [\mathbf{e}]_\times \mathbf{x}$, and $\mathbf{x}'$ lies on this line since $\mathbf{x}'^\top [\mathbf{e}]_\times \mathbf{x} = 0$.

- That is $\mathbf{x}$, $\mathbf{x}'$ and $\mathbf{e} = \mathbf{e}'$ are collinear (assuming both images are overlaid on top of each other).

- This collinearity property is termed auto-epipolar, and **does not hold** for general motion.

Special Motion: Pure Translation

Example 1:

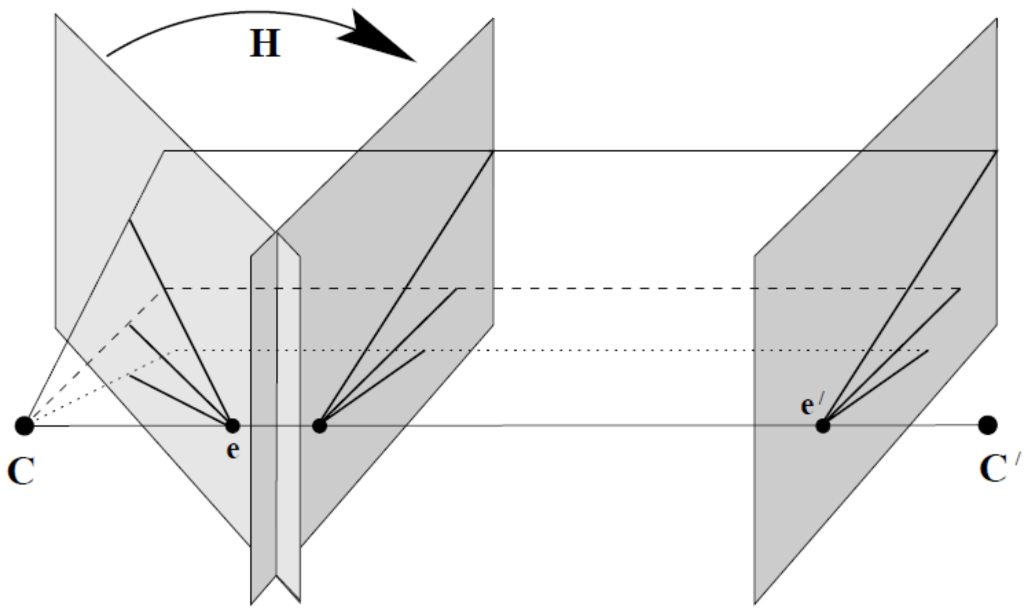


- We may consider the equivalent situation of pure translation.
- Camera is stationary, and the world undergoes a translation $-\mathbf{t}$.
- 3D points appear to slide along parallel rails.
- The images of these parallel lines intersect in a vanishing point corresponding to the translation direction.
- The epipole $\mathbf{e}$ is the vanishing point.

Source: Page 246, R. Hartley and A. Zisserman, "Multiple View Geometry in Computer Vision"

General Motion

- A general motion and its effect on the fundamental matrix can be separate into a pure rotation followed by a pure translation.



Source: Page 246, R. Hartley and A. Zisserman, "Multiple View Geometry in Computer Vision"

General Motion

- Now the the two cameras are given by $\mathrm{P} = \mathrm{K}[\mathrm{I} \mid \mathbf{0}]$ and $\mathrm{P} = \mathrm{K}[\mathrm{R} \mid \mathbf{t}]$.

- The pure rotation may be simulated by: $\mathtt{H} = \mathtt{K}'\mathtt{R}\mathtt{K}^{-1} = \mathtt{H}_{\infty}$, where H_{∞} is the infinite homography.

- As seen earlier, the fundamental matrix $\tilde{\mathrm{F}}$ under pure translation is given by $\tilde{\mathrm{F}} = [\mathbf{e}']_{\times}$.

- Since $\mathrm{F} = [\mathbf{e}']_{\times}\mathrm{K}'\mathrm{R}\mathrm{K}^{-1}$ (c.f. algebraic derivation), we have $\mathrm{F} = \tilde{\mathrm{F}}\mathrm{H}_{\infty} = [\mathbf{e}']_{\times}\mathrm{H}_{\infty}$.

Source: Page 246, R. Hartley and A. Zisserman, "Multiple View Geometry in Computer Vision"

Retrieving the Camera Matrices

- To this point we have examined the properties of F and of image relations for a point correspondence $\mathbf{x} \leftrightarrow \mathbf{x}'$.

- We now turn to one of the most significant properties of F, that the matrix may be used to determine the camera matrices of the two views.

Projective Invariance

- The fundamental matrices corresponding to the pairs of camera matrices $(\mathrm{P}, \mathrm{P}')$ and $(\mathrm{PH}, \mathrm{P}'\mathrm{H})$ are the same.

- H is a 4×4 matrix representing a projective transformation of 3-space.

Proof:

- Observe that $\mathrm{P}\mathbf{X} = (\mathrm{PH})(\mathrm{H}^{-1}\mathbf{X})$, and similarly for P'.
- Thus, if $\mathbf{x} \leftrightarrow \mathbf{x}'$ are matched points with respect to the pair of cameras $(\mathrm{P}, \mathrm{P}')$, corresponding to a 3D point $\mathbf{X}$.
- Then they are also matched points with respect to the pair of cameras $(\mathrm{PH}, \mathrm{P}'\mathrm{H})$, corresponding to the point $\mathrm{H}^{-1}\mathbf{X}$.

$\square$

Projective Invariance

- Thus, although a pair of camera matrices $(\mathrm{P}, \mathrm{P}')$ uniquely determine a fundamental matrix F, the converse is not true.

- The fundamental matrix determines the pair of camera matrices at best up to right-multiplication by a 3D projective transformation.

Given: $(\mathrm{P}, \mathrm{P}')$ —Unique→ F,

Given: F —Not Unique→ $(\mathrm{P}, \mathrm{P}')$ or $(\mathrm{PH}, \mathrm{P}'\mathrm{H})$

Canonical Form of Camera Matrices

- The fundamental matrix corresponding to a pair of camera matrices $\mathrm{P} = [\mathrm{I} \mid \mathbf{0}]$ and $\mathrm{P}' = [\mathrm{M} \mid \mathbf{m}]$ is equal to $[\mathbf{m}]_\times \mathrm{M}$.

Proof:

$$\mathbf{e}' = \mathrm{P}'\mathrm{C} = [\,\mathrm{M} \mid \mathbf{m}\,][0,0,0,1]^\top = \mathbf{m}$$

$$\mathrm{F} = [\mathbf{e}']_\times \mathrm{P}'\mathrm{P}^+ = [\mathbf{m}]_\times\,[\mathrm{M} \mid \mathbf{m}] \begin{bmatrix} \mathrm{I}_{3\times3} \\ 0_{3\times1} \end{bmatrix} = [\mathbf{m}]_\times \mathrm{M},$$

where $\mathrm{P}^+ = \begin{bmatrix} \mathrm{I}_{3\times3} \\ 0_{3\times1} \end{bmatrix}$ since $\mathrm{PP}^+ = \mathrm{I}$.

□

Projective Ambiguity of Cameras Given F

Theorem:

- Let $(\mathrm{P}, \mathrm{P}')$ and $(\tilde{\mathrm{P}}, \tilde{\mathrm{P}}')$ be two pairs of camera matrices such that F is the fundamental matrix corresponding to each of these pairs.
- Then, there exists a non-singular 4×4 matrix H such that $\tilde{\mathrm{P}} = \mathrm{PH}$ and $\tilde{\mathrm{P}}' = \mathrm{P}'\mathrm{H}$.

Projective Ambiguity of Cameras Given F

Proof:

- Suppose that a given fundamental matrix F corresponds to two different pairs of camera matrices $(\mathrm{P}, \mathrm{P}')$ and $(\tilde{\mathrm{P}}, \tilde{\mathrm{P}}')$.

- And the two pair of camera matrices is in canonical form with $\mathrm{P} = \tilde{\mathrm{P}} = [\mathrm{I} \mid \mathbf{0}]$, $\mathrm{P}' = [\mathrm{A} \mid \mathbf{a}]$ and $\tilde{\mathrm{P}}' = [\widetilde{\mathrm{A}} \mid \tilde{\mathbf{a}}]$.

- According to result of canonical cameras earlier, the fundamental matrix may then be written $\mathrm{F} = [\mathbf{a}]_\times \mathrm{A} = [\tilde{\mathbf{a}}]_\times \widetilde{\mathrm{A}}$.

Projective Ambiguity of Cameras Given F

Proof (cont.):

- We will need the following lemma:

Lemma:

Suppose the rank 2 matrix F can be decomposed in two different ways as $\mathrm{F} = [\mathbf{a}]_\times \mathrm{A}$ and $\mathrm{F} = [\tilde{\mathbf{a}}]_\times \widetilde{\mathrm{A}}$;

then $\tilde{\mathbf{a}} = k\mathbf{a}$ and $\widetilde{\mathrm{A}} = k^{-1}(\mathrm{A} + \mathbf{a}\mathbf{v}^\top)$ for some non-zero constant k and 3-vector $\mathbf{v}$.

Projective Ambiguity of Cameras Given F

(Lemma) Proof:

First, note that $\mathbf{a}^\top \mathrm{F} = \mathbf{a}^\top [\mathbf{a}]_\times \mathrm{A} = \mathbf{0}$, and similarly, $\tilde{\mathbf{a}}^\top \mathrm{F} = \mathbf{0}$. Since F has rank 2, it follows that $\tilde{\mathbf{a}} = k\mathbf{a}$ as required.

Next, from $[\mathbf{a}]_\times \mathrm{A} = [\tilde{\mathbf{a}}]_\times \widetilde{\mathrm{A}}$, it follows that $[\mathbf{a}]_\times (k\widetilde{\mathrm{A}} - \mathrm{A}) = 0$, and so $k\widetilde{\mathrm{A}} - \mathrm{A} = \mathbf{a}\mathbf{v}^\top$ for some **v**. Hence, $\widetilde{\mathrm{A}} = k^{-1}(\mathrm{A} + \mathbf{a}\mathbf{v}^\top)$ as required.

$\square$

Projective Ambiguity of Cameras Given F

Proof (cont.):

- Applying this result to the two camera matrices P and $\tilde{\mathrm{P}}$ shows that $\mathrm{P}' = [\mathrm{A} \mid \mathbf{a}]$ and $\tilde{\mathrm{P}}' = [k^{-1}(\mathrm{A} + \mathbf{a}\mathbf{v}^\top) \mid k\mathbf{a}]$ if they are to generate the same F.

- Now let $\mathrm{H} = \begin{bmatrix} k^{-1}\mathrm{I} & \mathbf{0} \\ k^{-1}\mathbf{v}^\top & k \end{bmatrix}$, we then we can verify that $\mathrm{PH} = k^{-1}[\mathrm{I} \mid \mathbf{0}] = k^{-1}\tilde{\mathrm{P}}$.

Projective Ambiguity of Cameras Given F

Proof (cont.):

- And furthermore,

$$\mathtt{P}'\mathtt{H} = [\mathtt{A} \mid \mathbf{a}]\mathtt{H} = [k^{-1}(\mathtt{A} + \mathbf{a}\mathbf{v}^\mathsf{T}) \mid k\mathbf{a}] = [\tilde{\mathtt{A}} \mid \tilde{\mathbf{a}}] = \tilde{\mathtt{P}}'$$

so that the pairs P, P' and $\tilde{\mathrm{P}}$, $\tilde{\mathrm{P}}'$ are indeed projectively related.

□

Decomposition of F Matrix

- A non-zero matrix F is the fundamental matrix corresponding to a pair of camera matrices P and P' if and only if $\mathrm{P}'^{\top}\mathrm{F}\mathrm{P}$ is skew-symmetric.

Proof:

The condition that $\mathrm{P}'^{\top}\mathrm{F}\mathrm{P}$ is skew-symmetric is equivalent to $\mathbf{X}^{\top}\mathrm{P}'^{\top}\mathrm{F}\mathrm{P}\mathbf{X} = 0$ for all $\mathbf{X}$.

Setting $\mathbf{x}' = \mathrm{P}'\mathbf{X}$ and $\mathbf{x} = \mathrm{P}\mathbf{X}$, this is equivalent to $\mathbf{x}'^{\top}\mathrm{F}\mathbf{x} = 0$, which is the defining equation for the fundamental matrix.

$\square$

Decomposition of F Matrix

- The camera matrices corresponding to a fundamental matrix F may be chosen as $\mathrm{P} = [\mathrm{I} \mid \mathbf{0}]$ and $\mathrm{P}' = [[\mathbf{e}']_\times \mathrm{F} \mid \mathbf{e}']$.

Proof:

We may verify that

$$[\mathrm{SF} \mid \mathrm{e}']^\mathsf{T}\mathrm{F}[\mathrm{I} \mid 0] = \begin{bmatrix} \mathrm{F}^\mathsf{T}\mathrm{S}^\mathsf{T}\mathrm{F} & \mathbf{0} \\ \mathrm{e}'^\mathsf{T}\mathrm{F} & 0 \end{bmatrix} = \begin{bmatrix} \mathrm{F}^\mathsf{T}\mathrm{S}^\mathsf{T}\mathrm{F} & \mathbf{0} \\ \mathbf{0}^\mathsf{T} & 0 \end{bmatrix}, \text{ where } \mathrm{S} = [\mathbf{e}']_\times.$$

which is skew-symmetric and hence F is a valid fundamental matrix (as we have proven previously).

□

Decomposition of F Matrix

- According to the Lemma seen earlier:

  F can be decomposed in two different ways as $\mathrm{F} = [\mathbf{a}]_\times \mathrm{A}$ and $\mathrm{F} = [\tilde{\mathbf{a}}]_\times \widetilde{\mathrm{A}}$, where $\tilde{\mathbf{a}} = k\mathbf{a}$ and $\widetilde{\mathrm{A}} = k^{-1}(\mathrm{A} + \mathbf{a}\mathbf{v}^\top)$ for some non-zero constant k and 3-vector **v**.

- The general formula for a pair of canonic camera matrices corresponding to a fundamental matrix F is given by:

$$\mathrm{P} = [\mathrm{I} \mid \mathbf{0}] \quad \mathrm{P}' = [[\mathbf{e}']_\times \mathrm{F} + \mathbf{e}'\mathbf{v}^\top \mid \lambda \mathbf{e}']$$

  where **v** is any 3-vector, and λ a non-zero scalar.

Essential Matrix

- **Normalized coordinates:** Known calibration matrices K and K' $\Rightarrow$ we can write $\mathbf{x} \leftrightarrow \mathbf{x}'$ as $\mathrm{K}^{-1}\mathbf{x} \leftrightarrow \mathrm{K}'^{-1}\mathbf{x}'$, i.e. $\hat{\mathbf{x}} \longleftrightarrow \hat{\mathbf{x}}'$:

$$\mathbf{x}'^{\top}\mathrm{F}\mathbf{x} = \mathbf{x}'^{\top}\mathrm{K}'^{-\top}\mathrm{E}\mathrm{K}^{-1}\mathbf{x} = 0$$

$$\hat{\mathbf{x}}'^{\top}\mathrm{E}\hat{\mathbf{x}} = 0$$

$$\hat{\mathbf{x}}'^{\top}[\mathbf{t}]_{\times}\mathrm{R}\hat{\mathbf{x}} = 0$$

- E is the Essential Matrix which can be expressed in terms of the relative transformation between two image frames.

Essential Matrix

Proof:

Previously we seen $\mathrm{F} = \lfloor \mathbf{e}' \rfloor_\times \mathrm{P}'\mathrm{P}^+$, since $\mathrm{P} = \mathrm{K}[\mathrm{I} \mid 0]$ and $\mathrm{P}' = \mathrm{K}'[\mathrm{R} \mid \mathbf{t}]$, we have:

$$\mathrm{P}^+ = \begin{bmatrix} \mathrm{K}^{-1} \\ 0_{1\times 3} \end{bmatrix}, \qquad \mathbf{C} = \begin{bmatrix} 0_{3\times 1} \\ 1 \end{bmatrix}$$

and

$$\mathrm{F} = \lfloor \mathbf{e}' \rfloor_\times \mathrm{P}'\mathrm{P}^+ = \lfloor \mathrm{P}'\mathbf{C} \rfloor_\times \mathrm{P}'\mathrm{P}^+$$

$$= \lfloor \mathrm{K}'\mathbf{t} \rfloor_\times \mathrm{K}'\mathrm{R}\mathrm{K}^{-1} = \mathrm{K}'^{-\mathsf{T}} \lfloor \mathrm{t} \rfloor_\times \mathrm{R}\mathrm{K}^{-1}$$

Properties of the Essential Matrix

- Five degree of freedom (3+3-1):
  - R and **t** have 3 degree of freedom each
  - But there is an overall scale ambiguity ⇒ -1 dof

- Singular values:
  - A 3 x 3 matrix is an essential matrix iff two of its singular values are equal, and the third is zero

Decomposition of E Matrix

- Extract R and t from the essential matrix E.

$$E = \lfloor \mathbf{t} \rfloor_\times R$$

Let us factorize $\lfloor \mathbf{t} \rfloor_\times$ and R into:

$$E = \lfloor t \rfloor_\times R = \underbrace{(UZU^\top)}_{\text{Skew-symmetric matrix}} \underbrace{(UXV^\top)}_{\text{Some rotation matrix}} = \underbrace{U(ZX)V^\top}_{\text{SVD of E}}$$

Since E is known up to a scale and ignoring the sign, we can set:

$$ZX = \begin{cases} ZW = \text{diag}(1,1,0) \\ ZW^\top = \text{diag}(-1,-1,0) \end{cases}, \quad \text{where} \quad Z = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad W = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Decomposition of E Matrix

- U and V are known from SVD of E.

Recovery of t: $[\mathbf{t}]_{\times} = \mathrm{UZU}^{\top}$

Since U is orthogonal and $[\mathbf{t}]_{\times}$ is skew-symmetric, we get:

$$\mathbf{t} = \pm\mathbf{U}_3, \quad \text{i.e. third column of U}$$

Recovery of R: $\mathrm{R} = \mathrm{UWV}^{\top}$, or $\mathrm{R} = \mathrm{UW}^{\top}\mathrm{V}^{\top}$

Make sure that R is in the Right-Hand Coordinate:

If det(R) < 0, then R = -R.

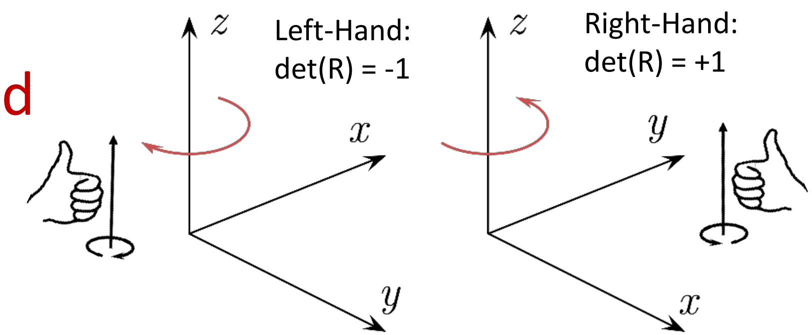
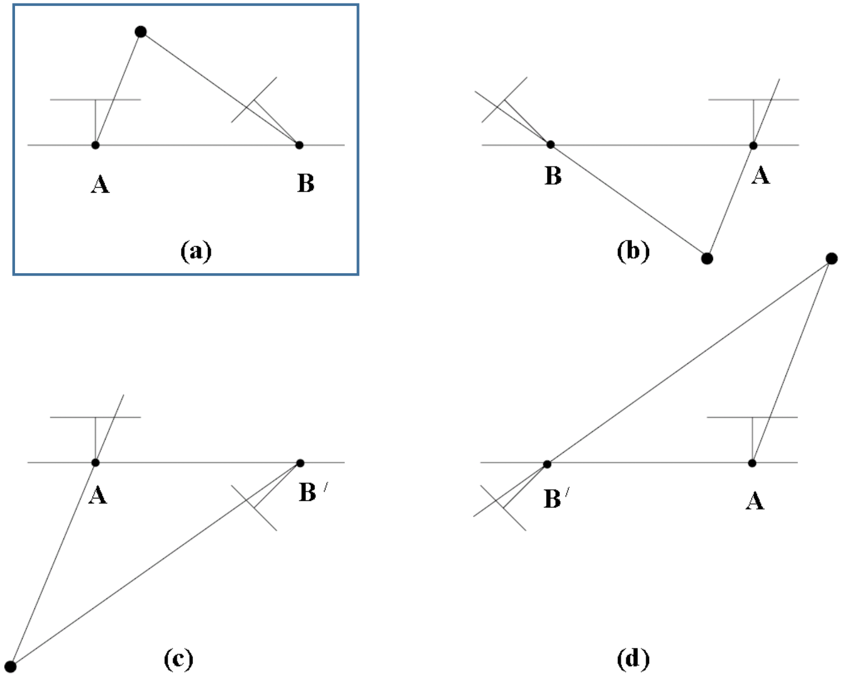


Image Source: https://en.wikipedia.org/wiki/Right-hand_rule

Decomposition of E Matrix

Four Possible Solutions for P':

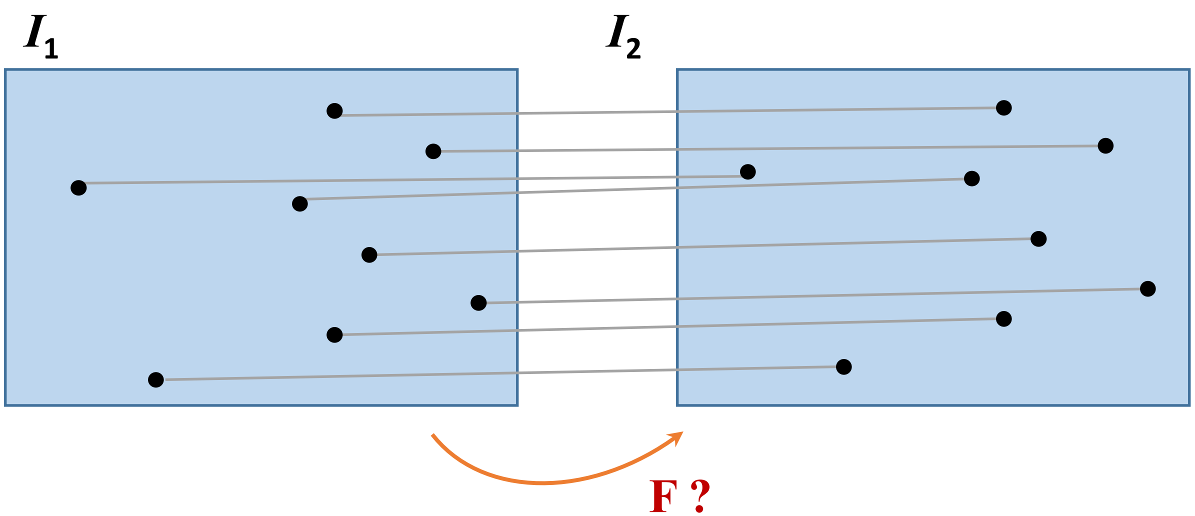
$$P' = [UWV^{\top} \mid +\mathbf{u}_3] \text{ or } [UWV^{\top} \mid -\mathbf{u}_3] \text{ or } [UW^{\top}V^{\top} \mid +\mathbf{u}_3] \text{ or } [UW^{\top}V^{\top} \mid -\mathbf{u}_3]$$



Only 1 of the 4 solutions is physically correct, i.e. the 3D point appears in front of both cameras.

Linear 8-Point Algorithm for F Matrix

- **Given**: A set of points correspondences $\mathbf{x}_i \leftrightarrow \mathbf{x}'_i$ between two images.
- **Compute**: The Fundamental matrix F.



Linear 8-Point Algorithm for F Matrix

- For any pair of matching points $\mathbf{x}_i \leftrightarrow \mathbf{x}_i'$ in two images, the 3x3 fundamental matrix is defined by the equation:

$$\mathbf{x}'^{\mathsf{T}}\mathrm{F}\mathbf{x} = 0$$

- Let $\mathbf{x} = (x, y, 1)^{\mathsf{T}}$ and $\mathbf{x}' = (x', y', 1)^{\mathsf{T}}$, we rewrite the above equation as:

$$x'xf_{11} + x'yf_{12} + x'f_{13} + y'xf_{21} + y'yf_{22} + y'f_{23} + xf_{31} + yf_{32} + f_{33} = 0$$

- Let **f** be the 9-vector made up of the entries of F in row-major order, we get:

$$(x'x, x'y, x', y'x, y'y, y', x, y, 1)\,\mathbf{f} = 0$$

Linear 8-Point Algorithm for F Matrix

- From a set of *n* point matches, we obtain a set of linear equations of the form:

$$\mathbf{A}\mathbf{f} = \begin{bmatrix} x'_1x_1 & x'_1y_1 & x'_1 & y'_1x_1 & y'_1y_1 & y'_1 & x_1 & y_1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ x'_nx_n & x'_ny_n & x'_n & y'_nx_n & y'_ny_n & y'_n & x_n & y_n & 1 \end{bmatrix} \mathbf{f} = \mathbf{0}$$

- A is a n x 9 matrix.
- For a non-trivial solution to exist, rank(A)=8 since f is a 9-vector.
- A minimum of 8-point correspondences is needed to solve for f.

Linear 8-Point Algorithm for F Matrix

- For noisy data, we obtain the solution of f by finding the least-squares solution.
- Least-squares solution for **f** is the singular vector corresponding to the smallest singular value of A.
- That is the last column of V in the SVD $A = UDV^T$.
- Similar to homography estimation, data normalization is needed.

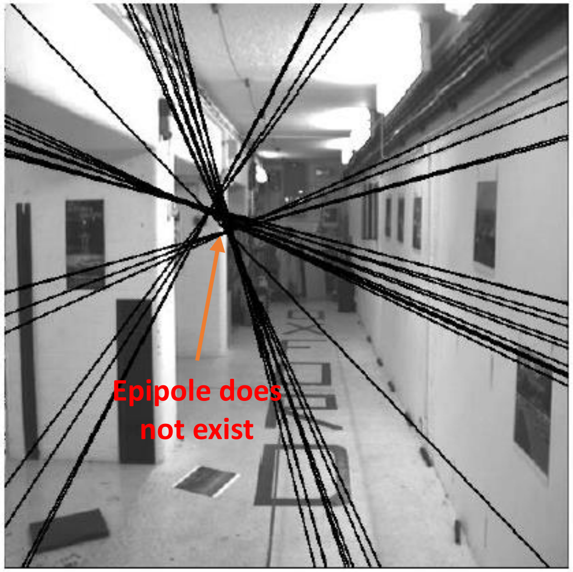
Singularity Constraint of F Matrix

- An important property of the fundamental matrix is that it is singular, i.e. rank(F) = 2.
- **Problem:** Least-squares solution in general will NOT give rank(F)=2.

Recall:

- Right and left nullspaces of F gives the epipoles, i.e. $\mathrm{F}\mathbf{e} = 0$, and $\mathrm{F}^\top\mathbf{e}' = 0$.
- Since e is a 3-vector, epipole exists if rank(F)=2.

$\mathrm{rank}(\mathrm{F}) \neq 2$



$\mathrm{rank}(\mathrm{F}) = 2$

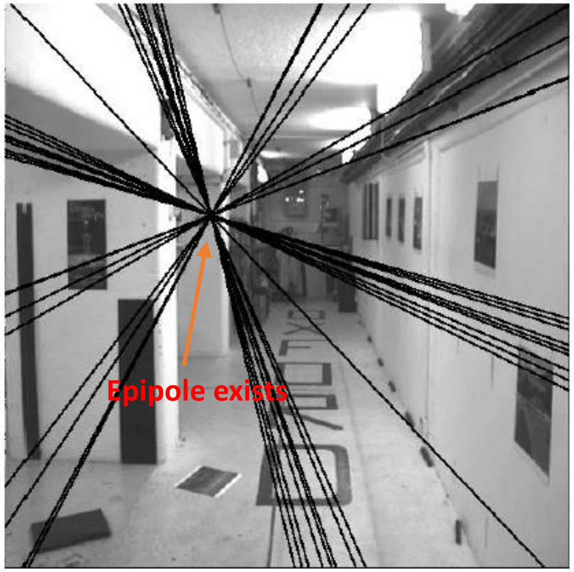


Image Source: R. Hartley and A. Zisserman, “Multiple View Geometry in Computer Vision”

Singularity Constraint of F Matrix

- Most convenient way is to correct the matrix F found by the SVD solution from A.
- Matrix F is replaced by the matrix F' that minimizes the Frobenius norm:

$$\min_{\mathrm{F}'} \|\mathrm{F} - \mathrm{F}'\|, \quad \text{s.t.} \qquad \det(\mathrm{F}') = 0$$

Steps:

1. Take SVD of F, i.e. $\mathrm{F} = \mathrm{UDV}^\top$, where $\mathrm{D} = \mathrm{diag}(r, s, t)$ satisfying $r \geq s \geq t$.
2. Then, $\mathrm{F}' = \mathrm{U}\mathrm{diag}(r, s, 0)\mathrm{V}^\top$ minimizes the Frobenius norm of $\mathrm{F} - \mathrm{F}'$.

Normalized 8-Point Algorithm for F Matrix

<u>Objective</u>

Given $n \geq 8$ image point correspondences $\{x_i \leftrightarrow x_i'\}$, compute the fundamental matrix F such that $\mathbf{x}_i'^{\top} \mathrm{F} \mathbf{x}_i = 0$.

<u>Algorithm</u>

(i) **Normalization:** Transform the image coordinates according to $\hat{\mathbf{x}}_i = \mathrm{T}\mathbf{x}_i$ and $\hat{\mathbf{x}}_i' = \mathrm{T}'\mathbf{x}_i'$, where T and T' are normalizing transformations.

(ii) Find the fundamental matrix $\hat{\mathrm{F}}'$ corresponding to the matches $\hat{\mathbf{x}}_i \leftrightarrow \hat{\mathbf{x}}_i'$ by:

- **a)** **Linear 8-point algorithm**.
- **b)** **Enforcing singularity constraint**.

(iii) Denormalization: Set $\mathrm{F} = \mathrm{T}'^{\top} \hat{\mathrm{F}}' \mathrm{T}$.

Note: RANSAC should be used for robust estimation!

$$\mathrm{T} = \begin{bmatrix} s & 0 & -sc_x \\ 0 & s & -sc_y \\ 0 & 0 & 1 \end{bmatrix}$$

c: centroid of all data points

$$s = \frac{\sqrt{2}}{\bar{d}}$$

where $\bar{d}$: mean distance of all points from centroid.

Normalized 8-Point Algorithm for E Matrix

<u>Objective</u>

Given $n \geq 8$ image point correspondences $\{x_i \leftrightarrow x_i'\}$ and the camera calibration matrices K and K', compute the essential matrix E such that $\mathbf{x}_i'^{\top}\, \mathrm{K}'^{-\top} \mathrm{E}\, \mathrm{K}^{-1} \mathbf{x}_i = 0$.

<u>Algorithm</u>

(i) **Normalized Coordinates:** For each correspondence $\mathbf{x}_i \leftrightarrow \mathbf{x}_i'$, compute $\mathrm{K}^{-1}\mathbf{x}_i \leftrightarrow \mathrm{K}'^{-1}\mathbf{x}_i'$, i.e. $\hat{\mathbf{x}} \leftrightarrow \hat{\mathbf{x}}'$.

(ii) Find the essential matrix E corresponding to the matches $\hat{\mathbf{x}}_i \leftrightarrow \hat{\mathbf{x}}_i'$ by:

- **a)** **Linear 8-point algorithm**.
- **b)** ***Enforcing singularity constraint**.

(iii) Decompose E to get R and **t**, thus forming P and P'.

*Singular constraint for E matrix is different from F matrix. See next slide for more detail.

Singularity Constraint of E Matrix

Problem:

- In general, the essential matrix E obtained from the linear 8-point algorithm will NOT have two similar singular values, and third is zero.

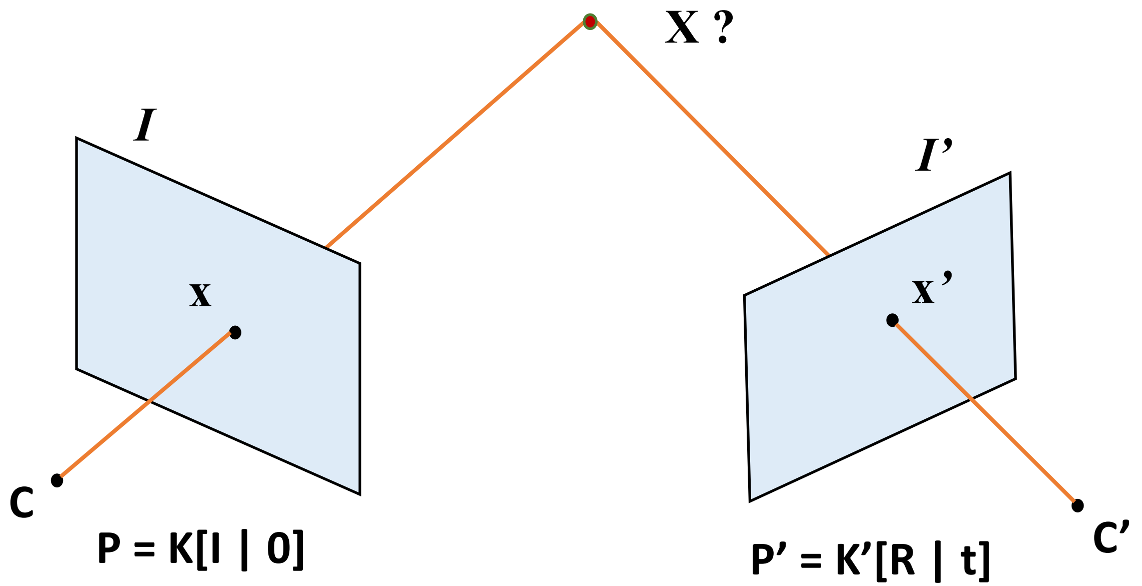
Solution:

1. Take SVD of E, i.e. $\mathrm{E} = \mathrm{UDV}^\top$, where $\mathrm{D} = \mathrm{diag}(a, b, c)$ with $a \geq b \geq c$.
2. The closest essential matrix to E in Frobenius norm is given $\hat{\mathrm{E}} = \mathrm{U}\hat{\mathrm{D}}\mathrm{V}^\top$, where

$$\hat{\mathrm{D}} = \mathrm{diag}(\frac{a+b}{2}, \frac{a+b}{2}, 0)$$

3D Structure Computation

- **Given:** The point correspondence $\mathbf{x}_i \leftrightarrow \mathbf{x}_i'$ and camera projection matrices P and P' of two images.
- **Find:** The 3D structure points $\mathbf{X}_i$ that corresponds to each 2D point correspondence.



3D Structure Computation

Linear Triangulation Method

- In each image, we have a measurement:

$$\mathbf{x} = \mathrm{P}\mathbf{X},\ \mathbf{x}' = \mathrm{P}'\mathbf{X}$$

- Unknown scale factor is eliminated by a cross-product, i.e. $\mathbf{x} \times (\mathrm{P}\mathbf{X}) = 0$ to give:

$$\begin{aligned} x(\mathbf{p}^{3\top}\mathbf{X}) - (\mathbf{p}^{1\top}\mathbf{X}) &= 0 \\ y(\mathbf{p}^{3\top}\mathbf{X}) - (\mathbf{p}^{2\top}\mathbf{X}) &= 0 \\ x(\mathbf{p}^{2\top}\mathbf{X}) - y(\mathbf{p}^{1\top}\mathbf{X}) &= 0 \end{aligned}$$

- $\mathrm{P}^{i\top}$ are rows of P.
- Two of the three equations are linearly independent.

3D Structure Computation

Linear Triangulation Method

- An equation of the form $\mathrm{A}\mathbf{X} = 0$ can be formed:

$$\mathtt{A} = \begin{bmatrix} x\mathbf{p}^{3\top} - \mathbf{p}^{1\top} \\ y\mathbf{p}^{3\top} - \mathbf{p}^{2\top} \\ x'\mathbf{p}'^{3\top} - \mathbf{p}'^{1\top} \\ y'\mathbf{p}'^{3\top} - \mathbf{p}'^{2\top} \end{bmatrix}$$

- Two equations from each image, giving a total of four equations in four homogeneous unknowns, i.e. $\mathbf{X} = [X\ Y\ Z\ 1]^{\top}$.

- Solution given by the right singular vector that corresponds to the smallest singular value of A, i.e. $\boldsymbol{v}_4$.

- $\mathbf{X} = \boldsymbol{v}_4 / \mathrm{v}_{4w} \Rightarrow$ to make last element of $\mathbf{X}$ equal to 1.

Reconstruction (Similarity) Ambiguity

- **Known Calibration:** Scene determined by the image is only up to a similarity transformation (rotation, translation and scaling).

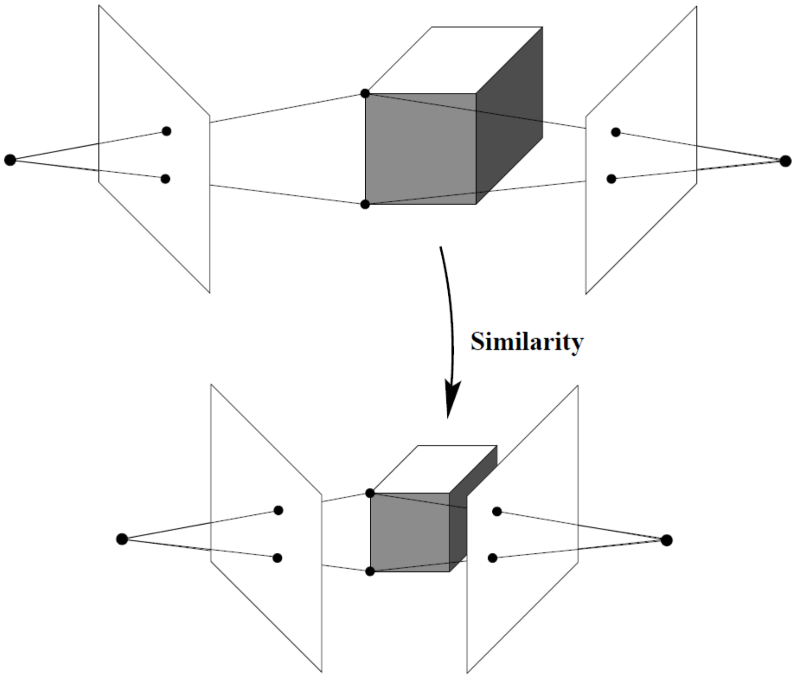


Image Source: R. Hartley and A. Zisserman, "Multiple View Geometry in Computer Vision"

Reconstruction (Similarity) Ambiguity

Proof sketch:

Let H_S be any similarity transformation: $H_S = \begin{bmatrix} R & \mathbf{t} \\ \mathbf{0}^\top & \lambda \end{bmatrix}$.

We can see that the projection on X_i is the same under P and PH_S^{-1}:

$$PX_i = (PH_S^{-1})(H_S X_i)$$

And PH_S^{-1} is still a valid projection matrix:

$$P = K[R_P \mid \mathbf{t}_P], \; PH_S^{-1} = K[R_P R^{-1} \mid \mathbf{t}']$$

□

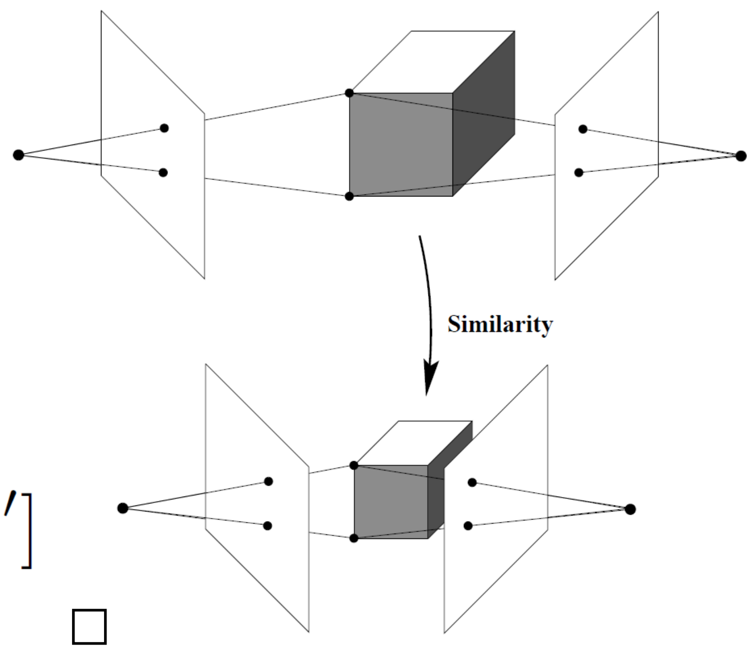


Image Source: R. Hartley and A. Zisserman, "Multiple View Geometry in Computer Vision"

Reconstruction (Projective) Ambiguity

- **Unknown Calibration:** We saw earlier that the fundamental matrix can be decomposed into P and P' or PH^{-1} and $\mathrm{P}'\mathrm{H}^{-1}$.

- The point $\mathbf{X}$ is reconstructed as $\mathrm{H}\mathbf{X}$ under PH^{-1} and $\mathrm{P}'\mathrm{H}^{-1}$ since:

$$\mathrm{P}\mathbf{X} = \mathrm{PH}^{-1}\mathrm{H}\mathbf{X}.$$

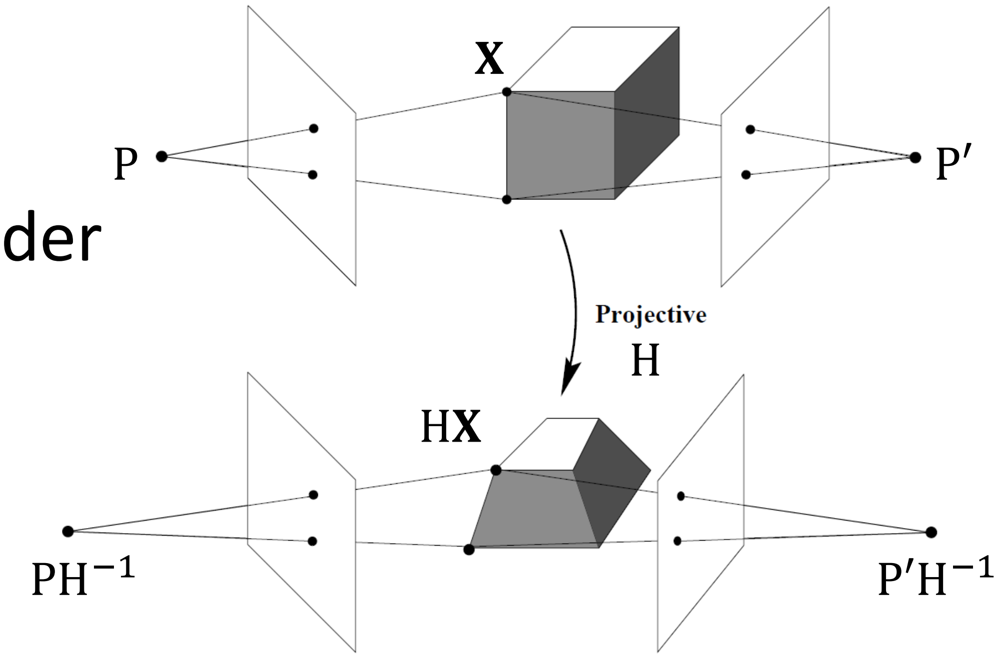


Image Source: R. Hartley and A. Zisserman, "Multiple View Geometry in Computer Vision"

Reconstruction (Projective) Ambiguity

- Original image pair


- Two different views of the reconstruction by P and P' decomposed from the F matrix obtained with the image pair.

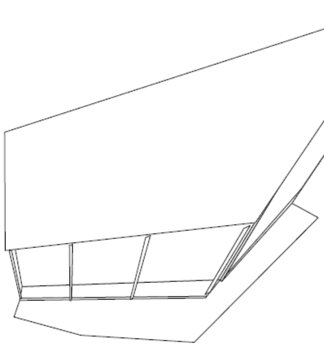
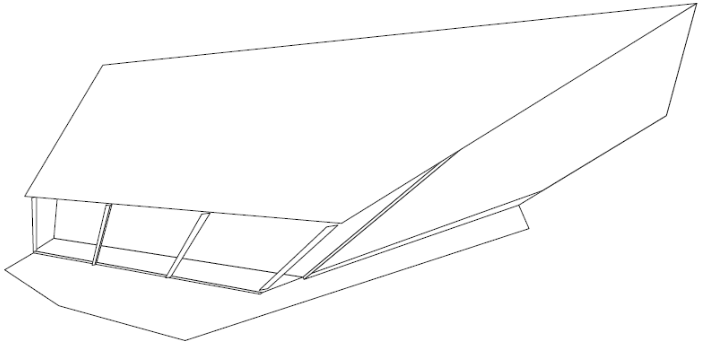



Image Source: R. Hartley and A. Zisserman, "Multiple View Geometry in Computer Vision"

Stratified Reconstruction

- The "stratified" approach to reconstruction:

1. Begin with a projective reconstruction.
2. And then refine it progressively to an affine.
3. Finally a metric reconstruction.

- We will see that affine and metric reconstruction are not possible without further information either about the scene, the motion or the camera calibration.

The Step to Affine Reconstruction

- The essence of affine reconstruction is to locate the plane at infinity.

- Let the 4-vector $\boldsymbol{\pi}$ be the plane at infinity under projective distortion; the goal is to find the projective transformation H that maps $\boldsymbol{\pi}$ to $(0,0,0,1)^\mathsf{T}$, i.e. $\boldsymbol{\pi}_\infty = \mathrm{H}^{-\mathsf{T}}\boldsymbol{\pi}$.

- H can be easily obtained as: $\mathrm{H} = \begin{bmatrix} \mathrm{I} \mid \mathbf{0} \\ \boldsymbol{\pi}^\mathsf{T} \end{bmatrix}.$

- Map all 3D reconstruction points X using H to remove the projective distortion (get an affine reconstruction).

The Step to Affine Reconstruction

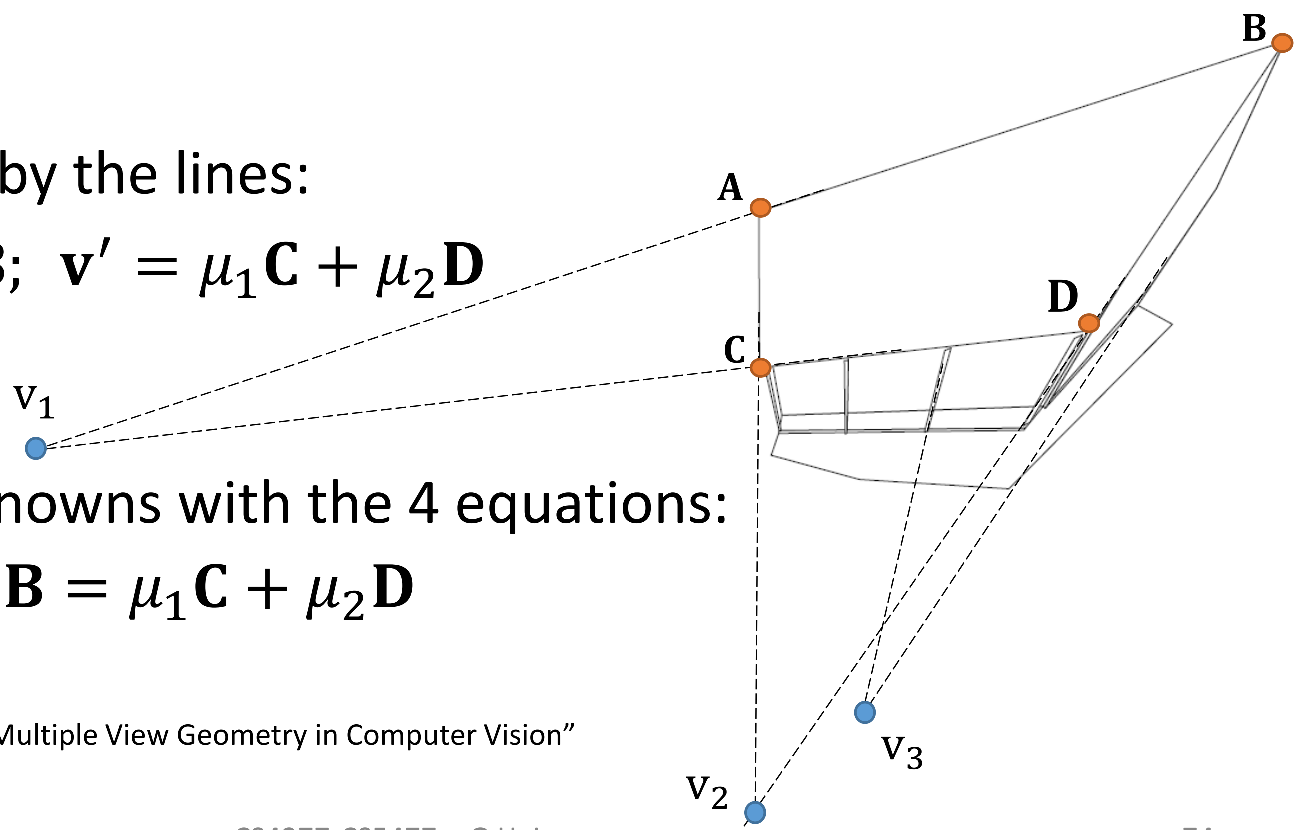
- Let $\mathrm{v}_1, \mathrm{v}_2, \mathrm{v}_3$ be the intersection points of a pair of parallel lines in three different directions, i.e. vanishing points.

- $\boldsymbol{\pi}$ can be identified from: $[\mathrm{v}_1 \quad \mathrm{v}_2 \quad \mathrm{v}_3]^\mathsf{T}\boldsymbol{\pi} = \mathbf{0}$.

Example:

Points on span by the lines:

$$\mathbf{v} = \lambda_1 \mathbf{A} + \lambda_2 \mathbf{B}; \;\; \mathbf{v}' = \mu_1 \mathbf{C} + \mu_2 \mathbf{D}$$



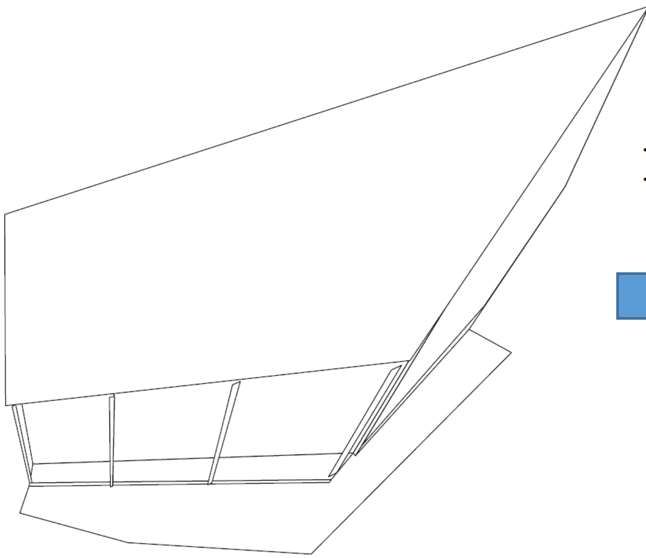
Solve for 4 unknowns with the 4 equations:

$$\mathbf{v}_1 = \lambda_1 \mathbf{A} + \lambda_2 \mathbf{B} = \mu_1 \mathbf{C} + \mu_2 \mathbf{D}$$

Image Source:
R. Hartley and A. Zisserman, “Multiple View Geometry in Computer Vision”

The Step to Affine Reconstruction

Projective distortion



$$H = \begin{bmatrix} I \mid \mathbf{0} \\ \boldsymbol{\pi}^T \end{bmatrix}$$

Affine distortion

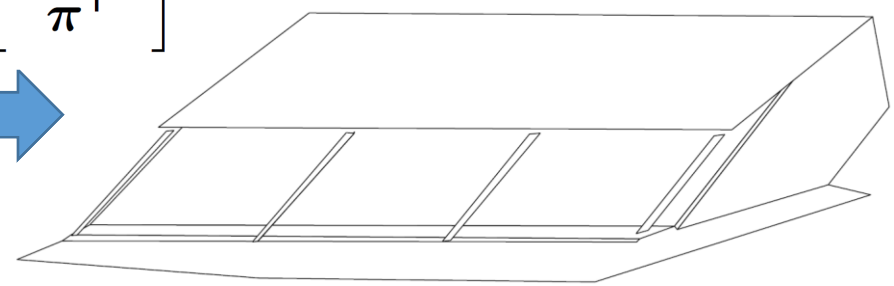


Image Source: R. Hartley and A. Zisserman, "Multiple View Geometry in Computer Vision"

The Step to Metric Reconstruction

- The key to metric reconstruction is the identification of the image of absolute conic ω (IAC).

- The affine reconstruction may be transformed to a metric reconstruction by applying a 3D transformation of the form:

$$\mathtt{H} = \begin{bmatrix} \mathtt{A}^{-1} & \\ & 1 \end{bmatrix},$$

where

- A is obtained by Cholesky factorization of $\mathtt{A}\mathtt{A}^{\mathsf{T}} = (\mathtt{M}^{\mathsf{T}}\omega\mathtt{M})^{-1}$.
- The affine reconstruction is from the camera matrix $\mathrm{P}' = [\mathrm{M} \mid \mathrm{m}]$.

The Step to Metric Reconstruction

Proof:

- We have seen earlier that under known calibration K', the camera matrix $\mathrm{P}'_\mathrm{M} = \mathrm{K}'[\mathrm{R} \mid \mathbf{t}]$ is subjected to similarity distortion.

- The affinely distorted camera matrix $\mathrm{P}' = [\mathrm{M} \mid \mathrm{m}]$ is transformed to P'_M as $\mathrm{P}'_\mathrm{M} = \mathrm{P}'\mathrm{H}^{-1}$, where

$$\mathrm{H}^{-1} = \begin{bmatrix} \mathrm{A} & \mathbf{0} \\ \mathbf{0}^\mathsf{T} & 1 \end{bmatrix} \quad \Rightarrow \quad [\,\mathrm{K}'\mathrm{R} \mid \mathrm{K}'\mathbf{t}\,] = [\mathrm{MA} \mid \mathrm{m}]$$

The Step to Metric Reconstruction

Proof (cont.):

- Hence, we get $\mathrm{MA} = \mathrm{K'R}$, which can be written as:

$$\mathrm{MA(MA)^\top = K'R(K'R)^\top} \Rightarrow \mathrm{MAA^\top M^\top = K'K'^\top}$$

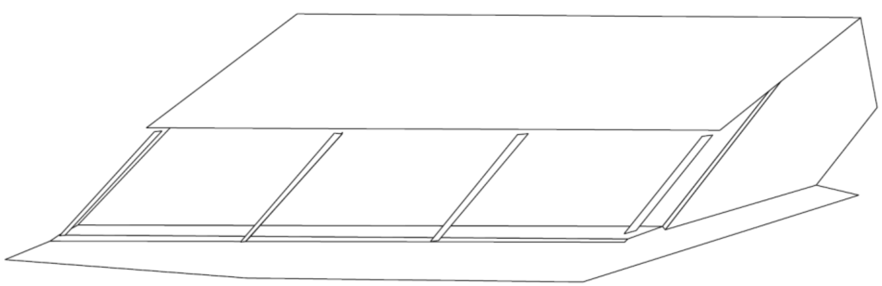
$$\Rightarrow \mathrm{AA^\top = M^{-1}}\underbrace{\mathrm{K'K'^\top}}_{\boldsymbol{\omega}^* = \boldsymbol{\omega}^{-1}}\mathrm{M^{-\top}}$$

$$\Rightarrow \mathrm{AA^\top = (M^\top}\boldsymbol{\omega}\mathrm{M)^{-1}}. \qquad \square$$

- Refer to Lecture 5 for the various methods to get the Image of absolute conic $\boldsymbol{\omega}$ (IAC).

The Step to Metric Reconstruction

Affine distortion



$$H = \begin{bmatrix} A^{-1} & \\ & 1 \end{bmatrix}$$

Similarity distortion

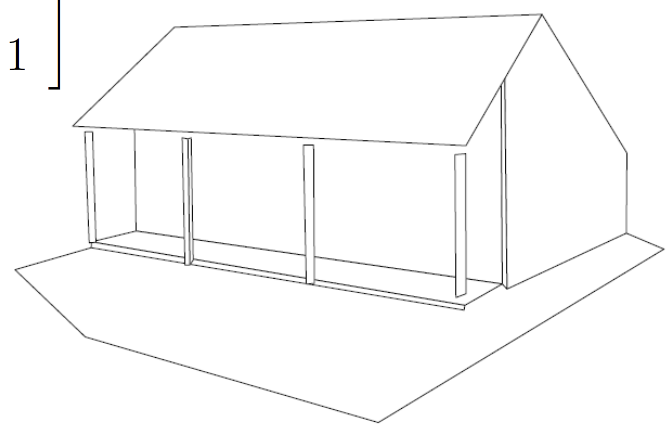


Image Source: R. Hartley and A. Zisserman, "Multiple View Geometry in Computer Vision"